Diagram groups

Burn Trip

Liam Stott

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Strings & free groups

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To consider a two-dimensional analogue of strings we need a two-dimensional analogue of a set - **a semigroup presentation** (a set + relations between strings).

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Similarly Δ has top path $\lceil \Delta \rceil$ and bottom path $\lfloor \Delta \rfloor$. If the top path of Δ has label u and the bottom path has label v then we say Δ is a (u, v)-diagram.

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In addition, for each string u we may consider the unique diagram ϵ_u with the property $\lceil \epsilon_u \rceil = \lfloor \epsilon_u \rfloor$ labelled by u.

This is known as the **trivial diagram** with label u and notice that for a (u, v)-diagram Δ we have $\epsilon_u \circ \Delta = \Delta \circ \epsilon_v = \Delta$.

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In this case we may obtain a diagram Δ' over $\mathcal{P}$ by deleting the path $\lfloor \pi_1 \rfloor = \lceil \pi_2 \rceil$ and identifying the paths $\lceil \pi_1 \rceil$ and $\lfloor \pi_2 \rfloor$. We then say that Δ' was obtained from Δ by **reducing a dipole**.

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Notice that a dipole in Δ is analogous to a cancelling inverse pair aa^{-1} in a string.

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We call $D(\mathcal{P}, u)$ the **diagram group** over $\mathcal{P}$ with base u.

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- $D(\mathcal{P}, x) \cong F$ where $\mathcal{P} = \langle x \mid x^2 = x \rangle$;
- $D(\mathcal{P}, x) \cong F_n$ where $\mathcal{P} = \langle x \mid x^n = x \rangle$.

Thompson's group F

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$$A(x) = \begin{cases} \frac{x}{2} & 0 \leq x \leq \frac{1}{2} \\ x - \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ 2x - 1 & \frac{3}{4} \leq x \leq 1 \end{cases}$$

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$$B(x) = \begin{cases} x & 0 \leq x \leq \frac{1}{2} \\ \frac{x}{2} + \frac{1}{4} & \frac{1}{2} \leq x \leq \frac{3}{4} \\ x - \frac{1}{8} & \frac{3}{4} \leq x \leq \frac{7}{8} \\ 2x - 1 & \frac{7}{8} \leq x \leq 1 \end{cases}$$