Analysis group intro

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Fractals

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- Fine or complicated detail at arbitrarily small scales.
- Self-similarity.
- Non-integer dimension.

Some examples

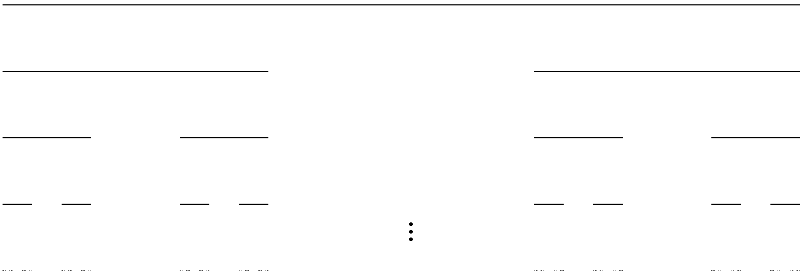

Figure: Middle third Cantor set

Some examples

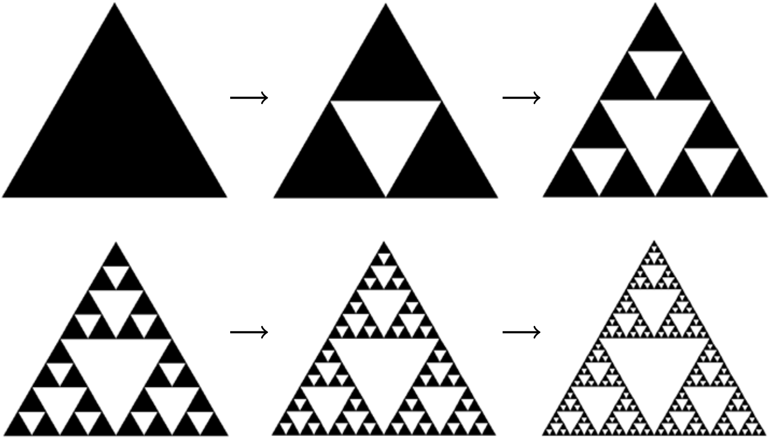

Figure: Sierpinski triangle

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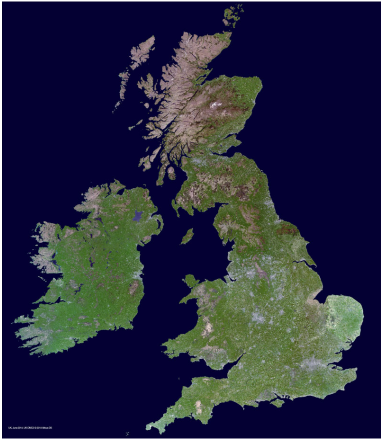

Figure: The coastline of the UK

Dimension theory

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To make this idea a bit more rigorous, it can be useful to think of these sets as having some 'mass' associated to them, and asking how this mass changes as you scale the object.

Dimension theory

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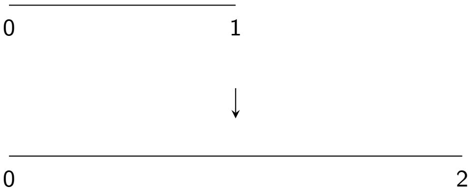

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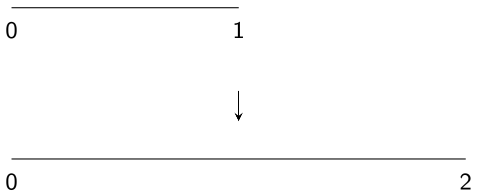

So scaling by a factor of 2 scales the 'mass' by a factor of 2^1.

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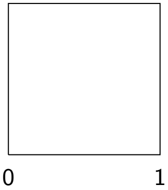

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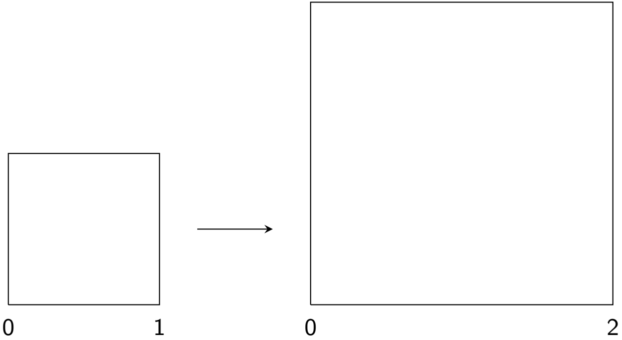

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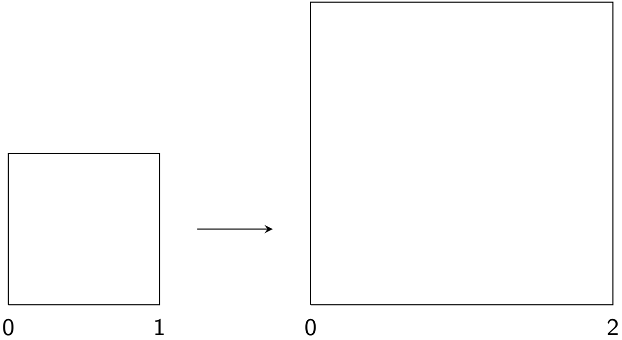

So scaling by a factor of 2 scales the 'mass' by a factor of 2^2.

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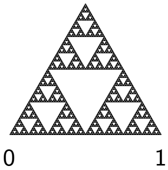

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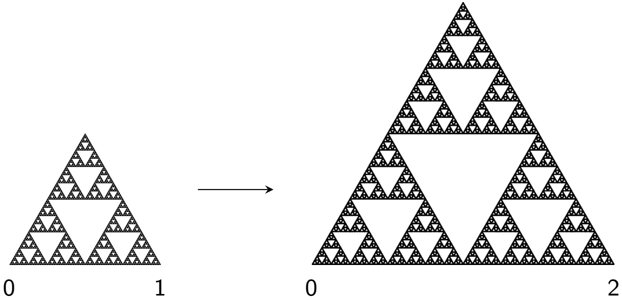

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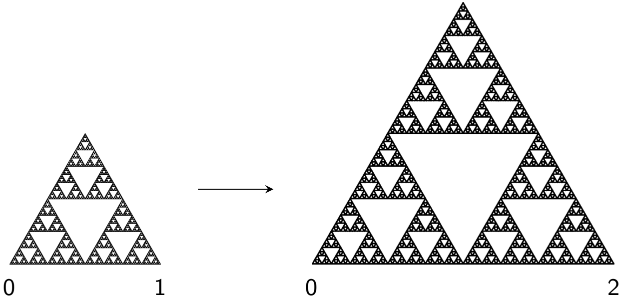

So scaling by a factor of 2 scales the 'mass' by a factor of $3 = 2^{\log 3/\log 2}$.

Hausdorff measure

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Hausdorff measure

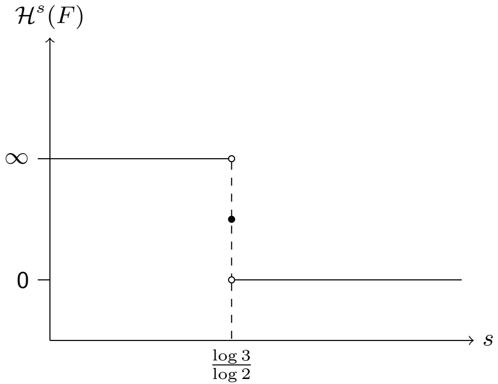


Figure: A plot of $\mathcal{H}^s(F)$ against s where F is the Sierpiński triangle.

Hausdorff measure

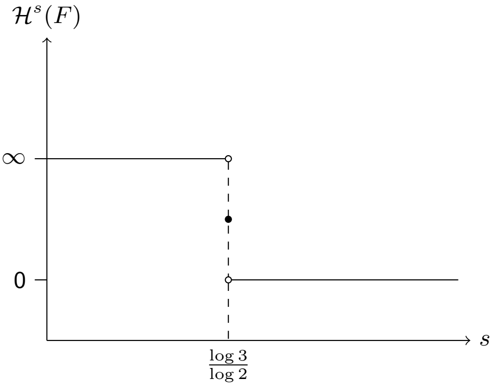


Figure: A plot of $\mathcal{H}^s(F)$ against s where F is the Sierpiński triangle.

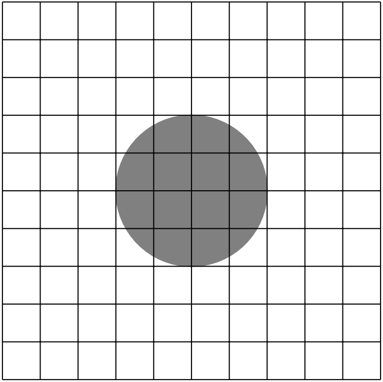
This value where the jump occurs is called the Hausdorff dimension of F, written as $\dim_{\mathrm{H}} F$.

Beyond the self-similar case

The issue with the approach used on the Sierpiński triangle is that fractals need not be self-similar. For example, the aforementioned coastline of a country will be far more erratic in nature compared to these 'nice' self-similar sets. So how can we get a handle on the dimensions of more complicated fractal sets?

Box counting dimension

Let's consider a disc lying in $\mathbb{R}^2$, and overlay a grid of boxes.



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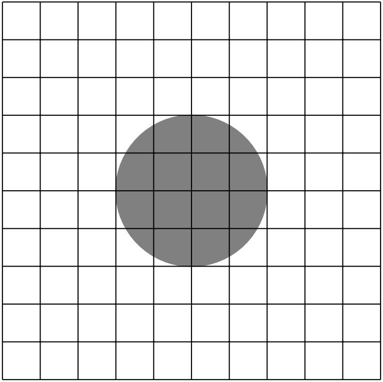
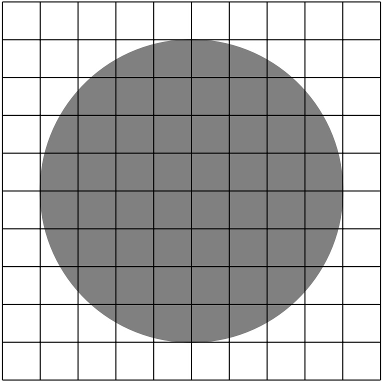


We have the intuition that scaling by a factor of s should scale the measure of the object by a factor of s^{dim} where dim is the dimension.

Box counting dimension

Therefore, suppose we double the radius of the disc.



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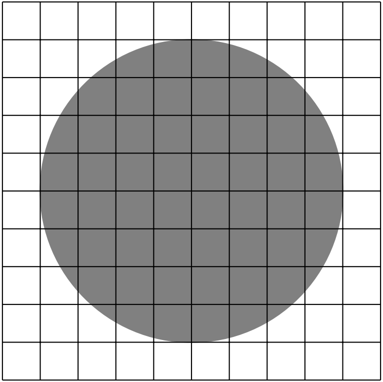

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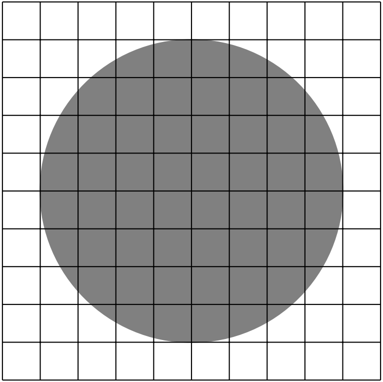

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Box counting dimension

More generally, suppose for a set F we were to plot the number of boxes touched $N(F)$ against the scaling factor s. The graph of best fit should take the form $N(F) = Cs^{\mathsf{dim(F)}}$.

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Box counting dimension

More formally, the box dimension of F is defined as

$$\mathsf{dim}_{\mathsf{B}} F = \lim_{r \to 0} \frac{\log N_r(F)}{-\log r}$$

where $N_r(F)$ denotes the smallest number of boxes of side length r required to cover F.

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So if $\mathsf{dim}_{\mathsf{B}} F = d$, then $N_r(F) \approx r^{-d}$.

Dynamical systems and fractals

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In particular, this chaotic behaviour means fractals can tend to show up naturally.

The Mandelbrot set

As an example, fix $c \in \mathbb{C}$ and consider the map $f_c(z) = z^2 + c$. Suppose we start at the point 0 and repeatedly apply the map $f_c(z)$.

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$$

What happens to $(f_c)^n(0)$ as $n \to \infty$?

The Mandelbrot set

It's possible that $(f_c)^n(0) \to \infty$ as $n \to \infty$, e.g. when $c = 1$

$$\begin{aligned} f_c(0) &= 1 \\ (f_c)^2(0) &= 1^2 + 1 = 2 \\ (f_c)^3(0) &= 2^2 + 1 = 5 \end{aligned}$$

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It's also possible that $(f_c)^n(0)$ remains bounded, e.g. when $c = -1$

$$\begin{aligned} f_c(0) &= -1 \\ (f_c)^2(0) &= (-1)^2 - 1 = 0 \\ (f_c)^3(0) &= 0^2 - 1 = -1. \end{aligned}$$

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We define the Mandelbrot set M as

$$M = \{c \in \mathbb{C} \mid (f_c)^n(0) \not\to \infty\}.$$

The Mandelbrot set

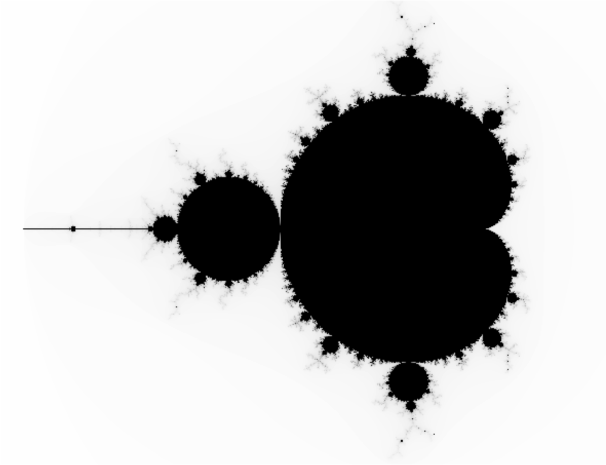

Figure: The Mandelbrot set

Mandelbrot set

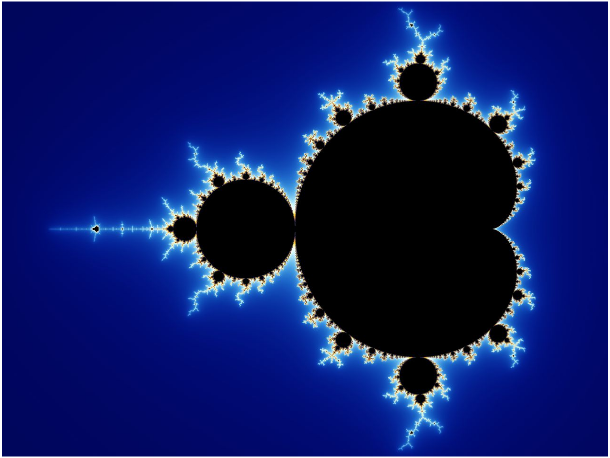

Figure: The Mandelbrot set (again)

Brownian motion

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Let $0 < \alpha < 1$. Fractional Brownian motion of index-α is defined to be a Gaussian process $B_\alpha : [0, \infty) \to \mathbb{R}$ such that

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- Almost surely, $B_\alpha(0) = 0$ and B_α is continuous.
- The increments are normally distributed with

$$B_\alpha(t) - B_\alpha(s) \sim \mathcal{N}(0, |s - t|^{2\alpha}).$$

Brownian motion

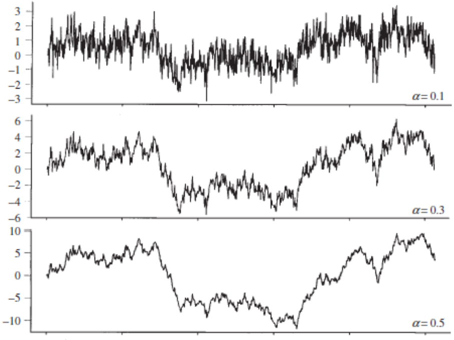

Figure: Some graphs of index-α Brownian motion. Taken from 'Fractal Geometry: Mathematical Foundations and Applications' by Kenneth Falconer.

Thank you for listening!

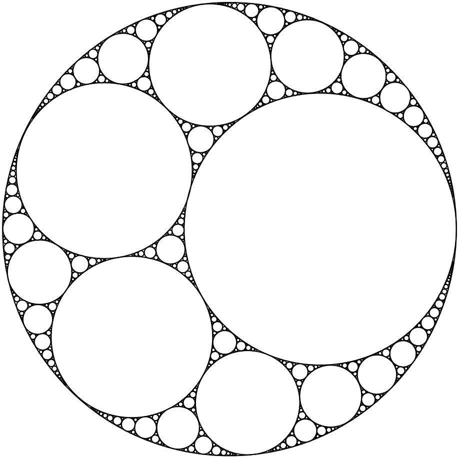

Figure: Apollonian gasket