Assoud dimension and self-similar sets satisfying the weak separation condition

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ABSTRACT. We discuss the weak separation condition in the context of iterated function systems of similarities in the real line. We then present a simplified proof of the Assouad dimension dichotomy result for self-similar subsets of the real line, originally due to Fraser–Henderson–Olson–Robinson. In particular, we show that if a self-similar set in $\mathbb{R}$ has a defining IFS which satisfies the weak separation condition, then the Assouad dimension agrees with the Hausdorff dimension; otherwise, the Assouad dimension is 1. We conclude with a discussion of generalizations of these results to higher dimensions.

CONTENTS

1. INTRODUCTION

One aspect of fractal geometry concerns the dimensional properties of subsets of the real line. There are a number of classical ways to understand the dimension of these sets, such as the Hausdorff and box dimensions (c.f. [Fal14]). In this document, we will focus on the Assouad dimension. Let $E \subseteq \mathbb{R}$ be a bounded Borel subset and for any $\rho > 0$ let $N_\rho(E)$ denote the smallest number of open balls

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Available online. https://rutar.org/notes/assouad_dichotomy.pdf with radius ρ required to cover E. We then define

$$N_{r,\rho}(E) = \sup_{x \in E} N_\rho(E \cap B(x,r))$$

where $B(x,r)$ is the open ball with radius r centred at x. Then the *Assouad dimension* of E, denoted by $\dim_{\mathrm{A}} E$, is given by

$$\dim_{\mathrm{A}} E = \inf\bigl\{s : \exists\, R_s, K_s \text{ s.t. } N_{r,\rho}(E) \leq K_s\,(r/\rho)^s \text{ for all } 0 < \rho < r \leq R_s\bigr\}.$$

The Assouad dimension was studied by Assouad [Ass77; Ass79] in order to study bi-Lipschitz embeddings of metric spaces. The following relationships are known or straightforward to prove. Let $\dim_{\mathrm{H}} E$ denote the Hausdorff dimension, and $\overline{\dim}_{\mathrm{B}} E$ and $\underline{\dim}_{\mathrm{B}} E$ denote the upper and lower box dimensions respectively. Then

$$\dim_{\mathrm{H}} E \leq \underline{\dim}_{\mathrm{B}} E \leq \overline{\dim}_{\mathrm{B}} E \leq \dim_{\mathrm{A}} E \leq 1. \tag{1.1}$$

It is also known that these inequalities may hold strictly. However, it is a question of interest to determine conditions under which equality may hold.

In this document, we will address this question in the context of self-similar subsets of the real line. Such sets are very important since they are simple to describe and construct, yet they still have many interesting properties and are poorly understood in general.

1.1. Iterated function systems and the weak separation condition. Fix a finite index set $\mathcal{I}$; then an iterated function system (IFS) of similarities, in $\mathbb{R}$, is a family of maps $\{S_i\}_{i\in\mathcal{I}}$ where $S_i(x) = r_i x + d_i : \mathbb{R} \to \mathbb{R}$ with $0 < |r_i| < 1$ for each $i \in \mathcal{I}$. To any IFS there exists a unique compact set $K \subset \mathbb{R}$ satisfying

$$K = \bigcup_{i\in\mathcal{I}} S_i(K),$$

which is referred to as the *self-similar set* of the IFS. In particular, a set $E \subseteq \mathbb{R}$ is said to be *self-similar* if there exists an IFS $\{S_i\}_{i\in\mathcal{I}}$ such that E is the self-similar set corresponding to the IFS.

If K is a singleton, then trivially equality holds in (1.1). Thus we assume that K is not a singleton; up to a normalization of the form $T \circ S_i \circ T^{-1}$ for some fixed similarity T, we may assume that the convex hull of K is the interval $[0,1]$. In general, it is known for self-similar subsets of $\mathbb{R}$ that $\dim_{\mathrm{H}} E = \underline{\dim}_{\mathrm{B}} E = \overline{\dim}_{\mathrm{B}} E$ [Fal14]. However, the relationship between the Hausdorff dimension and the Assouad dimension is more complicated and equality need not hold.

It turns out that the equality $\dim_{\mathrm{H}} E = \dim_{\mathrm{A}} E$ is governed by the *weak separation condition*. This notion was introduced by Lau & Ngai [LN99] and was designed as a generalization of the open set condition to allow more complicated iterated function systems with exact overlaps. Lau & Ngai used this notion to study dynamical properties of associated self-similar measures, while Bandt & Graf, and independently Zerner, investigated a different version of the definition, later proven to be equivalent, in order to study the dimensional properties of self-similar sets of IFSs satisfying the weak separation condition [BG92; Zer96].

2. ON THE WEAK SEPARATION CONDITION

In order to fully define the weak separation condition, we must introduce some additional notation. Let $\{S_i\}_{i\in\mathcal{I}}$ be an iterated function system of similarities, and let $\mathcal{I}^*$ denote the set of all finite words on $\mathcal{I}$. Given some word $\sigma=(i_1,\dots,i_n)\in\mathcal{I}^*$ so that each $i_j\in\mathcal{I}$, set

$$S_\sigma = S_{i_1}\circ\cdots\circ S_{i_n}, r_\sigma = r_{i_1}\cdots r_{i_n}, \text{ and } \sigma^- = (i_1,\dots,i_{n-1}).$$

We then set

$$\Lambda_\alpha = \{\sigma\in\mathcal{I}^* : |r_\sigma| < \alpha \leq |r_{\sigma^-}|\}$$

where, intuitively, Λ_α denotes the set of all words σ such that the corresponding function S_σ has contraction ratio approximately α. Given a set X, let $\#X$ denote the cardinality of X.

Definition 2.1 ([LN99]). We say that the IFS $\{S_i\}_{i\in\mathcal{I}}$ satisfies the *weak separation condition* if there exists some $x_0\in\mathbb{R}$ and $N\in\mathbb{N}$ such that for all $\sigma\in\mathcal{I}^*$, $\alpha\in(0,1)$, and $x\in K$,

$$\#\big(B(x,\alpha)\cap\{S_\omega(S_\sigma(x_0)) : \omega\in\Lambda_\alpha\}\big)\leq N.$$

We can prove a characterization of the weak separation condition in terms of compositions of functions. Let

$$\mathcal{E} = \{S_\sigma^{-1}\circ S_\tau : \sigma,\tau\in\mathcal{I}^*, \sigma\neq\tau\}$$

where $\mathcal{E}$ is a subset of the set of all similarities on $\mathbb{R}$, equipped with the topology of pointwise convergence. Note that the topology of pointwise convergence on the space of similarities is given by the topology of uniform convergence on K (when K is not a singleton). In particular, for f a similarity, denote $\|f\|_\infty = \sup_{x\in K}|f(x)|$.

2.1. Characterizing the weak separation condition.

We have the following result due to Zerner [Zer96] and Bandt & Graf [BG92]. The proof given is new and takes advantage of the straightforward geometry in $\mathbb{R}$.

Theorem 2.2. *Let $\{S_i\}_{i\in\mathcal{I}}$ be an IFS of similarities with self-similar set K not a singleton. Then $\{S_i\}_{i\in\mathcal{I}}$ satisfies the weak separation condition if and only if $\mathrm{Id}\notin\overline{\mathcal{E}\setminus\{\mathrm{Id}\}}$.*

Proof. ($\Longrightarrow$) Suppose for contradiction $\{S_i\}_{i\in\mathcal{I}}$ satisfies the weak separation condition and $\mathrm{Id}\in\overline{\mathcal{E}\setminus\{\mathrm{Id}\}}$. Let N be minimal such that Definition 2.1 holds and get some $x\in K$, $x_0\in K$, $\alpha>0$, distinct $S_{\omega_1},\dots,S_{\omega_N}$ with $\omega_i\in\Lambda_\alpha$, and $\xi\in\mathcal{I}^*$ such that $S_{\omega_i}(S_\xi(x_0))\in B(x,\alpha)$ for each i. Let

$$\epsilon_1 = \frac{\alpha/|r_{\omega_1}|-1}{2} \text{ and}$$
$$\epsilon_2 = \min\{\big\|S_{\omega_i}\circ S_{\omega_1}^{-1}-\mathrm{Id}\big\|_\infty : i\neq 1\}.$$

Note that $\epsilon_2 > 0$ since $S_{\omega_i} \neq S_{\omega_1}$ for $i \neq 1$. Finally, with $y = S_\xi(x_0)$, let $\delta > 0$ be such that $S_{\omega_1}(y) \in B(x, \alpha - \delta)$, set

$$\epsilon_3 = \frac{\delta}{2\alpha - \delta + 1}$$

and get $0 < \epsilon < \min\{\epsilon_1, \epsilon_2, \epsilon_3\}$. The choice of each ϵ_i corresponds directly to point (i) in Claim I for each $i = 1, 2, 3$.

Since $\mathrm{Id} \in \overline{\mathcal{E} \setminus \{\mathrm{Id}\}}$, get $\sigma, \tau \in \mathcal{I}^*$ with $S_\sigma \neq S_\tau$ such that $\|S_\sigma^{-1} \circ S_\tau - \mathrm{Id}\|_\infty < \epsilon$. Note that

$$\begin{aligned} |r_\sigma - r_\tau| &= |S_\sigma(1) - S_\sigma(0) + S_\tau(1) - S_\tau(0)| \leq 2\,\|S_\tau - S_\sigma\|_\infty \\ &\leq 2|r_\sigma|\,\left\|S_\sigma^{-1} \circ S_\tau - \mathrm{Id}\right\|_\infty < 2|r_\sigma|\epsilon \end{aligned}$$

so that

$$1 - 2\epsilon < \frac{r_\tau}{r_\sigma} < 1 + 2\epsilon \tag{2.1}$$

and thus $\|S_\tau^{-1} \circ S_\sigma - \mathrm{Id}\|_\infty < \frac{\epsilon}{1-2\epsilon}$. Therefore we may choose σ, τ such that

$$\left\|S_\sigma^{-1} \circ S_\tau - \mathrm{Id}\right\|_\infty < \epsilon \qquad \text{and} \qquad \left\|S_\tau^{-1} \circ S_\sigma - \mathrm{Id}\right\|_\infty < \epsilon.$$

In particular, we may assume without loss of generality that $|r_\sigma| \leq |r_\tau|$.

Now consider the words $\{\sigma\omega_1, \ldots, \sigma\omega_N\} \cup \{\tau\omega_1\}$. Recall that $y = S_\xi(x_0)$ from above.

Claim I. *From the choice of ϵ above, the following hold:*

1. *each $\sigma\omega_i$ and $\tau\omega_1$ are in $\Lambda_{|r_\sigma|\alpha}$,*
2. *the functions $S_{\sigma\omega_1}, \ldots, S_{\sigma\omega_N}, S_{\tau\omega_1}$ are distinct, and*
3. *$S_{\sigma\omega_i}(y) \in B(S_\sigma(x), |r_\sigma|\alpha)$ for each i and $S_{\tau\omega_1}(y) \in B(S_\sigma(x), |r_\sigma|\alpha)$.*

Assuming this, it is clear that the functions $S_{\sigma\omega_1}, \ldots, S_{\sigma\omega_N}, S_{\tau\omega_1}$ contradict the minimality of N, and we have the desired result.

Proof (of claim). We first see (1). Since the $\omega_i \in \Lambda_\alpha$, it is immediate that $\sigma\omega_i \in \Lambda_{|r_\sigma|\alpha}$. Since $|r_\sigma| \leq |r_\tau|$, we also have $|r_\sigma|\alpha \leq |r_{\sigma\omega_1^-}| \leq |r_{\tau\omega_1^-}|$. Thus it remains to show that $|r_{\tau\omega_1}| \leq |r_\sigma|\alpha$, or equivalently that $|r_\tau/r_\sigma| \leq \alpha/|r_\omega|$. But this follows directly by choice of $\epsilon < \epsilon_1$ and the estimate (2.1).

We now see (2). Since $S_\sigma \neq S_\tau$, we have $S_{\sigma\omega_1} \neq S_{\tau\omega_1}$. Otherwise for $i \neq 1$, suppose for contradiction $S_\sigma \circ S_{\omega_i} = S_\tau \circ S_{\omega_1}$. Then $S_\sigma^{-1} \circ S_\tau - \mathrm{Id} = S_{\omega_i} \circ S_{\omega_1}^{-1} - \mathrm{Id}$ but $\|S_\sigma^{-1} \circ S_\tau - \mathrm{Id}\|_\infty < \epsilon \leq \epsilon_2$ while $\left\|S_{\omega_i} \circ S_{\omega_1}^{-1} - \mathrm{Id}\right\| \geq \epsilon_2$ by choice of ϵ_2, a contradiction.

Finally, we see (3). Clearly $S_{\sigma\omega_i}(y) \in B(S_\sigma(x), |r_\sigma|\alpha)$ since

$$S_\sigma(B(x, \alpha)) = B(S_\sigma(x), |r_\sigma|\alpha).$$

Note that by choice of ϵ_3, since $\epsilon < \epsilon_3$, we have that $(1 + 2\epsilon)(\alpha - \delta) + \epsilon < \alpha$. Thus by applying (2.1)

$$|S_{\tau\omega_1}(y) - S_\sigma(x)| \leq |S_\tau(S_{\omega_1}(y)) - S_\tau(x)| + |S_\tau(x) - S_\sigma(x)|$$

$$\le |r_\tau|(\alpha-\delta)+|r_\sigma|\epsilon$$
$$\le |r_\sigma|\big((1+2\epsilon)\cdot(\alpha-\delta)+\epsilon\big) < |r_\sigma|\alpha$$

so that $S_{\tau\omega_1}(y)\in B(S_\sigma(x),|r_\sigma|\alpha)$, as claimed.

($\Longleftarrow$) The reverse direction is not needed for this document; for a proof, see [Zer96, Theorem 1]. The idea of the argument is that if the weak separation condition fails, then for any $M\in\mathbb{N}$, there is some ball $B(x,\alpha)$ and M distinct maps $S_{\sigma_1},\dots,S_{\sigma_M}$ such $S_\sigma(y)\in B(x,\alpha)$ for some y (depending, perhaps, on M). But then a Ramsey theorem argument along with an application of the pigeonhole principle applied to the values of the S_{σ_i} on two distinct points in K guarantees that for large M, some pair S_{σ_i}, S_{σ_j} must have $\left\|S_{\sigma_i}^{-1}\circ S_{\sigma_j}-\mathrm{Id}\right\|_\infty$ small. □

2.2. A uniform variation of the weak separation condition. The following result is useful since it essentially states that, locally, $B(x,\alpha)\cap K$ can be covered by some bounded number of images of K under maps S_σ with contraction ratios $|r_\sigma|\le\alpha$.

This result is conceptually useful, and also has a practical application in §3.1.

Proposition 2.3. *Suppose $\{S_i\}_{i\in\mathcal{I}}$ satisfies the weak separation condition. Then there exists some $M\in\mathbb{N}$ such that*

$$\sup_{x\in K}\#\{\sigma\in\Lambda_\alpha : B(x,\alpha)\cap S_\sigma(K)\neq\emptyset\}\le M.$$

Proof. For convenience assume the convex hull of K is $[0,1]$. Get some $x_0\in\mathbb{R}$ with corresponding bound N as in Definition 2.1. Let $S_{\sigma_1},\dots,S_{\sigma_n}$ be distinct similarities with $S_{\sigma_i}(K)\cap B(x,\alpha)\neq\emptyset$ and $\sigma_i\in\Lambda_\alpha$. I claim that $n\le 2N$. For each $1\le i\le n$, since $S_{\sigma_i}(K)\cap B(x,\alpha)\neq\emptyset$, since $|S_{\sigma_i}(K)|\le\alpha$, there exists $z_i\in\{0,1\}$ such that $S_{\sigma_i}(z_0)\in B(x,\alpha)$. Thus get $\epsilon>0$ such that $B(S_{\sigma_i}(z_i),\epsilon)\subset B(x,\alpha)$. Since $K\subseteq\overline{\{S_\tau(x_0):\tau\in\mathcal{I}^*\}}$, get τ_0 such that $S_{\tau_0}(x_0)\in B(0,\epsilon)$ and τ_1 such that $S_{\tau_1}(x_0)\in B(1,\epsilon)$.

But then if i is such that $z_i=0$, then $S_{\sigma_i}(z_i)\in B(x,\alpha)$ so that $S_{\sigma_i}(S_{\tau_0}(x_0))\in B(x,\alpha)$ since S_{σ_i} is a contraction, and thus $\#\{i:z_i=0\}\le N$. Thus $n\le 2N$, as claimed. □

3. PROOF OF THE DICHOTOMY RESULT

In this section, we prove the main dichotomy result: if $K\subseteq\mathbb{R}$ is a self-similar set which is not a singleton, then $\dim_{\mathrm{H}}K=\dim_{\mathrm{A}}K$ if K satisfies the weak separation condition, and $\dim_{\mathrm{A}}K=1$ if K does not satisfy the weak separation condition. We separate this proof into two distinctions.

3.1. Self-similar sets with the weak separation condition. It is proven in, for example, [FHO+15] that if the defining IFS of K satisfies the weak separation condition, then K is in fact *Ahlfors regular*, which means that there are constants $a,b>0$ such that for any $x\in K$,

$$a\alpha^s\le\mathcal{H}^s(K\cap B(x,\alpha))\le b\alpha^s$$

where $\mathcal{H}^s(K)$ is the Hausdorff s-measure of K. The proof of this fact uses Proposition 2.3, as well as the fact that $0 < \mathcal{H}^s(K) < \infty$, which follows using a similar proof technique as in [Fal14, Theorem 3.1].

Here we will present a direct proof that under the weak separation condition that $\dim_{\mathrm{H}} K = \dim_{\mathrm{A}} K$. The proof is similar to that of Fraser in [Fra14, Theorem 2.10], but the idea is standard.

Theorem 3.1. *Suppose the IFS $\{S_i\}_{i\in\mathcal{I}}$ satisfies the weak separation condition with self-similar set K. Then $\dim_{\mathrm{H}} K = \dim_{\mathrm{A}} K$.*

Proof. Recall that $\dim_{\mathrm{H}} K = \overline{\dim}_{\mathrm{B}} K$ since K is a self-similar set. Set $s = \overline{\dim}_{\mathrm{B}} K$ and let $\epsilon > 0$; we will show that $\dim_{\mathrm{A}} K \leq s + \epsilon$, from which the result follows.

Let $0 < \rho < r \leq |K|$ and $x \in K$ be arbitrary. Let M be a constant as in Proposition 2.3 and get maps $S_{\sigma_1}, \ldots, S_{\sigma_k}$ with $k \leq M$ and $\sigma_i \in \Lambda_\rho$ such that $B(x,\rho) \cap K \subseteq \bigcup_{i=1}^k S_{\sigma_i}(K)$. In particular, note that $|r_{\sigma_i}| \leq \rho < r$ for each $1 \leq i \leq k$.

By definition of the box dimension, get some constant C_ϵ such that for any $0 < R \leq 1$, $N_R(K) \leq C_\epsilon R^{-s-\epsilon}$. In particular, get some ρ/r cover of K given by $\{U_j\}_{j=1}^\ell$ where $\ell \leq C_\epsilon (r/\rho)^{s+\epsilon}$, so that $\bigcup_{i=1}^k \{S_{\sigma_i}(U_j)\}$ is a ρ-cover of $B(x,\rho) \cap K$ and thus

$$N_\rho(B(x,r) \cap K) \leq MC_\epsilon \left(\frac{r}{\rho}\right)^{s+\epsilon}.$$

But x was arbitrary so that $N_{r,\rho}(K) \leq MC_\epsilon (r/\rho)^{s+\epsilon}$, and therefore $\dim_{\mathrm{A}} K \leq s + \epsilon$, as required. □

3.2. Weak pseudo-tangents. Our goal is now to prove the second half of the dichotomy result: if K is a self-similar set that is not a singleton and the defining IFS does not satisfy the weak separation condition, then $\dim_{\mathrm{A}} K = 1$. The main mechanism through which we will do this is to construct a *weak pseudo-tangent*. The notion of a weak pseudo-tangent is a modification of the idea of a weak tangent, developed by Mackay & Tyson [MT10] which is in turn based on the notion of a *microset* due to Furstenberg.

Denote the Hausdorff pseudo-metric $p_H(X,Y) = \sup_{x\in X} \inf_{y\in Y} |x-y|$.

Definition 3.2. Let F and $\widehat{F}$ be compact subsets of $\mathbb{R}^d$. We say that $\widehat{F}$ is a weak pseudo-tangent of F if there exists a sequence of similarities $T_k : \mathbb{R} \to \mathbb{R}$ such that $p_H(\widehat{F}, T_k(F)) \to 0$ as $k \to \infty$.

Proposition 3.3. *If $\widehat{F}$ is a weak pseudo-tangent of F, then $\dim_{\mathrm{A}} \widehat{F} \leq \dim_{\mathrm{A}} F$.*

Proof. Recall that the Assouad dimension is preserved under similarities. The idea behind this proof is to use the maps T_k to move covers of F to covers of $T_k(F)$; since $p_H(\widehat{F}, T_k(F)) \to 0$, these covers can be arbitrarily good covers of $\widehat{F}$.

Let $s > \dim_{\mathrm{A}} F$ be arbitrary. Since F is compact (hence bounded), there exists a constant K_s such that for any $0 < \rho < r \leq 1$, $N_{r,\rho}(F) \leq K_s (r/\rho)^s$. Get similarities T_k satisfying the definition of the weak pseudo-tangent Definition 3.2. Since the T_k are similarities, if T_k has contraction ratio u_k, then

$$N_{r,\rho}(T_k(F)) = N_{r/u_k,\rho/u_k}(F) \leq K_s (r/\rho)^s$$

as well for any $0 < \rho < r \leq 1$, where K_s does not depend on K. Let k be sufficiently large so that $p_H(\widehat{F}, T_k(F)) \leq \rho/4$, and thus

$$\widehat{F} \subseteq \bigcup_{y \in T_k(F)} B(y, \rho/2). \tag{3.1}$$

Now given $x \in \widehat{F}$, construct a cover for $\widehat{F} \cap B(x, r)$ as follows. Let $y \in T_k(F)$ have $|x - y| < \rho/2$ so that $B(x, r) \subseteq B(y, 2r)$ since $\rho < r$. Then get a $\rho/2$-cover $\{B(y_i, \rho/2)\}_{i=1}^N$ for $T_k(F) \cap B_{2r}(y)$ where $N \leq K_s \left(\frac{2r}{\rho/2}\right)^s$, and thus applying (3.1)

$$\widehat{F} \cap B(x, r) \subseteq \bigcup_{i=1}^N \bigcup_{y \in B(y_i, \rho/2)} B(y, \rho/2) = \bigcup_{i=1}^N B(y_i, \rho)$$

so that $N_\rho(\widehat{F} \cap B(x, r)) \leq K_s 4^s (r/\rho)^s$ for all $x \in \widehat{F}$ with $0 < \rho < r < 1/2$. But $s > \dim_A F$ was arbitrary so $\dim_A \widehat{F} \leq \dim_A F$. □

3.3. Self-similar sets without the weak separation condition. We are now in position to prove our main result. This result was originally proven in [FHO+15, Theorem 1.3]. We give a modified version of the proof given in [AKT20, Theorem 4.1], with simplifications since the IFS is composed of similarities.

Theorem 3.4. *Let $\{S_i\}_{i \in \mathcal{I}}$ be an IFS not satisfying the weak separation condition with associated self-similar set K. If K is not a singleton, then $\dim_A K = 1$.*

Proof. Without loss of generality, we may assume that $0 \in K$ and there is $\mathbf{0} \in \mathcal{I}$ so that $S_{\mathbf{0}}(0) = 0$ and $r_{\mathbf{0}} > 0$. Since the weak separation condition fails, by Theorem 2.2 for every $\epsilon > 0$ there are words $\sigma, \tau \in \mathcal{I}^*$ so that

$$S_\sigma^{-1} \circ S_\tau(x) = \gamma x + \delta$$

for some $0 \leq \delta \leq \epsilon$ and $|1 - \gamma| \leq \epsilon$ with $(\delta, \gamma) \neq (0, 1)$. By appending at most two letters to σ and τ if necessary, we may assume that $r_\sigma > 0$, $r_\tau > 0$, and $\delta > 0$.

Now fix $m \in \mathbb{N}$. Using the above observation, inductively choose $\{\sigma_\ell, \tau_\ell, k_\ell\}$ for each $\ell = 1, \ldots, m$ so that with $\phi_\ell = \tau_{\ell-1}\mathbf{0}^{k_{\ell-1}} \cdots \tau_1 \mathbf{0}^{k_1}$ and $c_\ell = r_{\mathbf{0}}^{k_1 + \cdots + k_{\ell-1}} r_{\sigma_1} \cdots r_{\sigma_{\ell-1}}$ (taking ϕ_1 to be the empty word and $c_1 = 1$)

1. $S_{\sigma_\ell}^{-1} \circ S_{\tau_\ell}(x) = \gamma_\ell x + \delta_\ell$ where

$$0 < \delta_\ell \leq \frac{c_\ell}{m} \qquad \text{and} \qquad |S_{\phi_\ell}(0)(\gamma_\ell - 1)| \leq \frac{r_{\mathbf{0}} \cdot c_\ell}{2m},$$

   and

2. $k_\ell \in \mathbb{N} \cup \{0\}$ satisfies

$$\frac{r_{\mathbf{0}}}{m} < \frac{r_{\mathbf{0}}^{-k_\ell} \delta_\ell}{c_\ell} \leq \frac{1}{m}.$$

Now set $\psi_\ell = \sigma_m \mathbf{0}^{k_m} \cdots \sigma_\ell \mathbf{0}^{k_\ell}$ and write $\rho = r_{\mathbf{0}}^{k_1+\cdots+k_m} r_{\sigma_1} \cdots r_{\sigma_m}$. By construction,

$$S_{\mathbf{0}}^{-k_\ell} \circ S_{\sigma_\ell}^{-1} \circ S_{\tau_\ell} \circ S_{\mathbf{0}}^{k_\ell}(x) = \gamma_\ell x + \delta_\ell r_{\mathbf{0}}^{-k_\ell}$$

so that

$$\begin{aligned} S_{\psi_{\ell+1}\phi_{\ell+1}}(0) - S_{\psi_\ell \phi_\ell}(0) &= S_{\psi_\ell} \circ S_{\mathbf{0}}^{-k_\ell} \circ S_{\sigma_\ell}^{-1} \circ S_{\tau_\ell} \circ S_{\mathbf{0}}^{k_\ell} \circ S_{\phi_\ell}(0) - S_{\psi_\ell\phi_\ell}(0) \\ &= r_{\sigma_\ell} \cdots r_{\sigma_m} \cdot r_{\mathbf{0}}^{k_\ell+\cdots+k_m} (r_{\mathbf{0}}^{-k_\ell}\delta_\ell + S_{\phi_\ell}(0)(\gamma_\ell - 1)) \\ &= \rho \Delta_\ell \end{aligned}$$

for some Δ_ℓ satisfying $r_{\mathbf{0}}/2 < m\Delta_\ell \leq 1 + r_{\mathbf{0}}/2$ by the choice of k_ℓ. In particular,

$$\rho^{-1}(K - S_{\psi_1\phi_1}(0)) \supset \{0, \Delta_1, \ldots, \Delta_1 + \cdots + \Delta_{m-1}\}.$$

But m was arbitrary, so $[0,1]$ is a weak pseudo-tangent of K and $\dim_{\mathrm{A}} K = 1$. □

3.4. Generalizations to higher dimensions. The definitions presented above (of the weak separation condition, Assouad dimension, etc.) generalize naturally to higher dimensions. In addition, the characterizations proven in Theorem 2.2 and Proposition 2.3, as well as the Assouad dimension under the weak separation condition Theorem 3.1 can be shown to hold in higher dimensions as well. The general proof of Theorem 2.2 can be found in [Zer96], while the proofs of Proposition 2.3 and Theorem 3.1 generalize to higher dimensions with minimal modification.

However, the dichotomy result Theorem 3.4 does not hold strictly in higher dimensions; indeed, the best that one can obtain is the following:

Theorem 3.5 ([FHO+15]). *Let $K \subseteq \mathbb{R}^d$ be a self-similar set not contained in any $(d-1)$-dimensional hyperplane. If the defining IFS for K does not satisfy the weak separation condition, then $\dim_{\mathrm{A}} K \geq 1$.*

Certainly this result is sharp in $\mathbb{R}$, but it is also sharp in higher dimension. Consider the IFS on $[0,1]^2$ defined for $t \in [0,4]$ by the maps

$$S_1(x) = x/5 \qquad S_2(x) = x/5 + (t/5, 0)$$
$$S_3(x) = x/5 + (4/5, 0) \qquad S_4(x) = x/5 + (0, 4/5)$$

with self-similar set K has $\dim_{\mathrm{H}} K \leq \log(4)/\log(5) < 1$. But even if t is chosen so that the WSC fails, we have $1 \leq \dim_{\mathrm{A}} K \leq 1 + \log 2/\log 5$ since K is contained in the $[0,1] \times C$ where C is the Cantor set on the second coordinate axis formed by the maps $x/5$ and $x/5 + 4/5$. Of course, by changing the parameter 5 to some arbitrary $r \geq 5$, we see that the inequality $\dim_{\mathrm{A}} K \geq 1$ is in fact sharp.

Similarly, it is also possible to construct self-similar sets K in $\mathbb{R}^d$ with $\dim_{\mathrm{A}} K = d$ while $\dim_{\mathrm{H}} K$ can be made arbitrarily small. For details of this, we refer the reader to [FHO+15, §4.1 and §4.2], while more precise information can be found in the work of García [Gar20].

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