# Box versus packing dimensions via anti-Frostman measures

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ABSTRACT. We give a proof of the relationship between the upper box and packing dimensions using an "anti-Frostman" result for upper box dimension due to Falconer–Fraser–Käenmäki.

## 1. BOX VERSUS PACKING DIMENSIONS

**1.1. Upper box and packing dimensions of sets.** There are many notions of the "dimension" of a set, but perhaps one of the easiest notions of dimension to define in terms of covers is the *upper box dimension*. Throughout this document, we fix $d \in \mathbb{N}$ and work in $\mathbb{R}^d$. Then for $r > 0$, let $N_r(K)$ denote the least number of balls with radius $r$ required to cover $K$. In other words, $N_r(K)$ is the smallest number so that there are points $\{y_1, \ldots, y_{N_r(K)}\}$ so that

$$K \subset \bigcup_{i=1}^{N_r(K)} B(y_i, r).$$

We then write

$$\overline{\dim}_{\mathrm{B}} K = \limsup_{r \to 0} \frac{\log N_r(K)}{\log(1/r)}.$$

On the other hand, the *packing dimension* of a set $K$ is often defined through measures. Let $s \geq 0$ be fixed. We first define *$s$-dimensional packing pre-measure* for an arbitrary set $E \subset \mathbb{R}^d$ as

$$\mathcal{P}_0^s(E) = \lim_{\delta \to 0} \left\{ \sum_{i=1}^{\infty} B(x_i, r_i)^s : \begin{array}{c} \{B(x_i, r_i)\} \text{ pairwise disjoint} \\ \text{with } r_i \leq \delta \text{ and } x_i \in E \end{array} \right\}.$$

Note that the limit always exists by monotonicity. We call such a family $\{B(x_i, r_i)\}$ with $x_i \in E$ and the balls pairwise disjoint a *centred packing*. However, $\mathcal{P}_0^s(E)$ is not countably stable by considering, for example, any countable dense subset of $\mathbb{R}^d$. Countably stabilizing the measure yields *$s$-dimensional packing measure*:

$$\mathcal{P}^s(K) = \inf \left\{ \sum_{i=1}^{\infty} \mathcal{P}_0^s(E_i) : K \subset \bigcup_{i=1}^{\infty} E_i \right\}.$$

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Finally, similarly to the Hausdorff dimension, we define the *packing dimension* of $K$ as the critical exponent at which the packing measure jumps from zero to infinity.

$$\dim_{\mathrm{P}} K = \inf\{s \geq 0 : \mathcal{P}^s(K) = 0\}.$$

One downside of the box dimension is that it is not countably stable: in particular, for any bounded set $F$, $\overline{\dim}_{\mathrm{B}} F = \overline{\dim}_{\mathrm{B}} \overline{F}$. For example, by taking a countable dense subset of any set, one obtains a set with packing dimension zero but arbitrary upper box dimension. It turns out that in some sense this is the only distinction between upper box and packing dimension: the main goal of this document is to prove that the packing dimension is *countably stabilized* upper box dimension, i.e. for any bounded set $F$,

$$\dim_{\mathrm{P}} F = \inf\left\{\sup_j \overline{\dim}_{\mathrm{B}} E_j : F \subset \bigcup_{j=1}^{\infty} E_j\right\}.$$

The remainder of the document will be focused on a proof of this fact.

**1.2. An anti-Frostman lemma.** In this section, we prove a useful anti-Frostman lemma for the upper box dimension of a set $K \subset \mathbb{R}^d$. This lemma was first proven in [FFK23, Theorem 2.1] and we adapt the proof directly from this paper.

**Lemma 1.1.** *Let $K \subset \mathbb{R}^d$ be bounded. Then there is a probability measure $\mu$ with $\operatorname{supp}\mu = \overline{K}$ such that for any $\epsilon > 0$, $x \in \overline{K}$, and $r \in (0,1)$,*

$$\mu\big(B(x,r)\big) \gtrsim_\epsilon r^{\overline{\dim}_{\mathrm{B}} K + \epsilon}.$$

*Proof.* For each $n \in \mathbb{N}$, let $m_n := N_{2^{-n}}(K)$ and write $s_n = \frac{\log m_n}{n \log 2}$. Note that $\overline{\dim}_{\mathrm{B}} K = \limsup_{n\to\infty} s_n$.

Let's first construct for each $n \in \mathbb{N}$ a measure satisfying the correct properties at scale $2^{-n}$. First, get a maximal $2^{-n}$-separated set $E_n = \{x_{n,1}, \dots, x_{n,m_n}\} \subset K$. Let $\mu_n$ denote the uniform measure on $E_n$; in other words, $\mu_n$ is the sum of point masses on each $x_{i,n}$ each with weight $1/m_n$. Note that if $x \in K$ is arbitrary and $r \in [2^{-n}, 2^{-(n-1)})$, then

$$\mu_n\big(B(x,r)\big) \geq m_n^{-1} = (2^{-n})^{s_n} \gtrsim r^{s_n}. \tag{1.1}$$

While each measure $\mu_n$ has the correct properties at scale $n$, we wish to construct a measure which has the correct properties at all scales simultaneously. Let $\gamma = \sum_{n=1}^{\infty} \frac{1}{n^2}$ and let

$$\mu = \frac{1}{\gamma} \sum_{n=1}^{\infty} \frac{\mu_n}{n^2}.$$

Let's show that $\mu$ satisfies the correct properties. Since $\mu$ is a probability measure, it suffices to prove the result for all $r$ sufficiently small. Let $\epsilon > 0$ and let $x \in K$

and $r \in (0,1)$. Let $n \in \mathbb{N}$ be the unique choice so that $r \in [2^{-n}, 2^{-(n-1)})$. Now for $r$ sufficiently small, $s_n \leq \overline{\dim}_{\mathrm{B}} K + \epsilon/2$ and $\frac{1}{n^2} \leq r^{\epsilon/2}$ so that applying (1.1)

$$\begin{aligned}\mu\big(B(x,r)\big) &\geq \mu\big(B(x,r)\big)\\ &\gtrsim \frac{1}{n^2}\mu_n\big(B(x,r)\big)\\ &\gtrsim \frac{1}{n^2} r^{\overline{\dim}_{\mathrm{B}} K + \epsilon/2}\\ &\gtrsim r^{\overline{\dim}_{\mathrm{B}} K + \epsilon}.\end{aligned}$$

Finally, multiplying by an additional small constant depending on $\epsilon$ yields the result for all $r \in (0,1)$. □

**Remark 1.2.** In some sense, the main idea in the above proof is the opposite of dyadic pigeonholing: instead of using polynomial weights to recover a good scale at which to operate, we use polynomial weights to combine a collection of objects each of which is only relevant at a fixed scale. Of course, the cost of using such weights is that there is an additional sub-exponential error term in the estimates: but for our purposes, this is easily enough.

**Remark 1.3.** In [FFK23], the infimum over exponents $s$ for which $\mu\big(B(x,r)\big) \gtrsim_s r^s$ for all $r \in (0,1)$ is called the *upper Minkowski dimension* of the measure $\mu$. This term is used since the upper box dimension (also known as the upper Minkowski dimension) is the infimum over upper Minkowski dimensions of measures which are fully supported on $\mu$.

However, Frostman's lemma (and the dual mass distribution principle) also implies that the Hausdorff dimension of a set is the best exponent $s$ for which there exists a measure $\mu$ satisfying $\mu\big(B(x,r)\big) \lesssim_s r^s$. Note that this exponent is not, in general, equal to the usual definition of the Hausdorff dimension of a measure.

**1.3. Reducing upper box dimensions to packings.** In order to prove the equivalence of upper box and packing dimensions, we must first use the upper box dimension to make a statement about packings. First, let's observe an easy equivalent definition. For $r > 0$, let $P_r(K)$ denote the maximal cardinality of a $r$-separated subset of $K$. In other words, $P_r(K)$ is the largest number so that there are points $\{x_1, \ldots, x_{P_r(K)}\}$ such that $x_i \in K$ for all $i$ and $d(x_i, x_j) \geq r$, and if $x \in K$ is arbitrary, then there is some $j$ so that $d(x, x_j) < r$.

Any maximal $r$-separated subset must immediately yield a cover, since any point with distance at least $r$ from every point in the subset could be added, violating maximality. Therefore $N_r(K) \leq P_r(K)$. On the other hand, any ball $B(y,r)$ contains at most $1$ centre of a maximal $2r$-packing, so $P_r(K) \leq N_{r/2}(K)$. In particular,

$$\overline{\dim}_{\mathrm{B}} K = \limsup_{r \to 0} \frac{\log P_r(K)}{\log(1/r)}.$$

However, this reduction only permits balls of a fixed radius: in the definition of packing dimension, we need to handle arbitrary countable packings. We now make this reduction using the anti-Frostman lemma proven in the previous section.

**Lemma 1.4.** *Let $K \subset \mathbb{R}^d$ be bounded. Then $\overline{\dim}_{\mathrm{B}} K$ is the infimum over all numbers $s$ such that for any centred packing $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of $K$ with $r_i \in (0,1)$, $\sum_{i=1}^{\infty} r_i^s \lesssim_s 1$.*

In some sense, this lemma provides the motivation for why box and packing dimensions are equal. While the proof is an easy application of the anti-Frostman lemma from the previous section, the result itself is mildly counterintuitive. This proof also highlights the usefulness of constructing highly uniform measures supported on sets for which you wish to bound the dimension: in this situation, the measure exploits the disjointness of the balls in the cover in a very natural way.

*Proof.* By taking a centred packing in which $r_i = r/2$ for all $i$, we observe that $\overline{\dim}_{\mathrm{B}} \leq s$. Conversely, fix $\epsilon > 0$ and apply Lemma 1.1 to get a measure $\mu$ such $\mu\big(B(x,r)\big) \gtrsim r^{\overline{\dim}_{\mathrm{B}} K + \epsilon}$ for all $x \in K$ and $r \in (0,1)$. Then for any centred packing $\{B(x_i, r_i)\}_{i=1}^{\infty}$, by disjointness,

$$1 = \mu(K) \geq \sum_{i=1}^{\infty} \mu\big(B(x_i, r_i)\big) \gtrsim_\epsilon r_i^{\overline{\dim}_{\mathrm{B}} K + \epsilon}.$$

This gives the result. □

**Remark 1.5.** Note that Lemma 1.4 proves that the upper box dimension of $K$ is equal to what is sometimes called the *disk packing exponent* of $K$. More detailed discussion of this phenomenon, and other notions of dimension related the upper box dimension, can be found in [BP17, §2.6].

**1.4. Equivalence of packing and stabilized upper box dimensions.** To conclude, in this section we prove that packing dimension is countably stabilized upper box dimension. This result was perhaps first proven in [Tri82, Proposition 2].

We begin with a lemma directly relating pre-packing measure with the upper box dimension. Our main result will follow from this directly.

**Lemma 1.6.** *Let $K \subset \mathbb{R}^d$ be arbitrary and $s \geq 0$.*
*(i) If $\mathcal{P}_0^s(E) < \infty$, then $\overline{\dim}_{\mathrm{B}} E \leq s$.*
*(ii) If $\mathcal{P}_0^s(E) > 0$, then $\overline{\dim}_{\mathrm{B}} E \geq s$.*

*Proof.* To see (i), since $\mathcal{P}_0^s(E) < \infty$, there is an $\epsilon > 0$ and $M > 0$ so that any packing $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of $K$ with $0 < r_i \leq \epsilon$ has $\sum_{i=1}^{\infty} r_i^s \leq M$. In particular, taking a packing with all $r_i \leq r$, we directly see that $P_{2r}(E) \leq r^s = (2r)^s 2^{-s}$. This implies that $\overline{\dim}_{\mathrm{B}} E \leq s$.

Then to prove the contrapositive of (ii), write $s = \overline{\dim}_{\mathrm{B}} E$ and let $\delta > 0$. By Lemma 1.4 applied to the exponent $s + \delta/2$, for any centred packing $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of $K$ with all $0 < r_i \leq r$,

$$\sum_{i=1}^{\infty} r_i^{s+\delta} \lesssim_\delta r^{\delta/2}.$$

Taking the limit as $r$ converges to zero gives that $\mathcal{P}_0^{s+\delta}(E) = 0$, as required. □

Our main claim concerning packing dimension now follows directly.

**Theorem 1.7.** *Let $K \subset \mathbb{R}^d$ be arbitrary. Then*

$$\dim_{\mathrm{P}} K = \inf \left\{ \sup_j \overline{\dim}_{\mathrm{B}} E_j : K \subset \bigcup_{j=1}^{\infty} E_j \right\}.$$

*Proof.* First, suppose $\dim_{\mathrm{P}} K < s$. By definition, there is a family $\{E_j\}_{j=1}^{\infty}$ covering $K$ such that $\mathcal{P}_0^s(E_j) < \infty$ for all $i$. Moreover, by Lemma 1.6 (i), $\overline{\dim}_{\mathrm{B}} E_j \leq s$. Since $s > \dim_{\mathrm{P}} K$ was arbitrary, we conclude that

$$\dim_{\mathrm{P}} K \geq \inf \left\{ \sup_j \overline{\dim}_{\mathrm{B}} E_j : K \subset \bigcup_{j=1}^{\infty} E_j \right\}. \tag{1.2}$$

Now suppose for contradiction (1.2) holds with a strict inequality. Then there is a $s < \dim_{\mathrm{P}} K$ so that $K \subset \bigcup_{j=1}^{\infty} E_j$ where $\overline{\dim}_{\mathrm{B}} E_j \leq s$. But Lemma 1.6 (ii) implies that $\mathcal{P}_0^s(E_j) = 0$ so $\mathcal{P}^s(E) = 0$ and $\dim_{\mathrm{P}} K \leq s$, a contradiction. Thus the desired equality holds. □

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