The Kakeya conjecture in three dimensions

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ABSTRACT. We discuss the *Kakeya conjecture*, a question involving quantitative estimates for configurations of line segments in d-dimensional space. This conjecture was recently resolved by Hong Wang and Joshua Zahl in dimension 3. We also discuss the significance of this conjecture and its relationship with questions in harmonic analysis, PDEs, and other areas of mathematics farther afield.

The Kakeya needle problem. The original *Kakeya needle problem*, posed by Sōichi Kakeya in 1917 [Kak17], asks the following: what is the smallest area required to rotate a unit line segment (or a "needle") in the plane by continuous motion $360°$? A related question, but without requiring continuous motion, was independently asked and resolved by Besicovitch in 1920 [Bes20] in a Russian periodical not accessible to readers outside Russia.

For some time, it was thought that the optimal shape was a *deltoid*. However, in 1928, Besicovitch published a complete answer to this question: even requiring a continuous motion, the area can be made arbitrarily small [Bes28]. In fact, if one simply asks for a set containing a unit segment pointing in each direction, the area can even be zero. Such a set is typically called a *Kakeya* or *Besicovitch* set; this terminology is also sensible even beyond the plane.

Questions involving Kakeya sets seem, at first glance, to be geometric curiosities. However, starting in the 1970s, many remarkable and deep connections to questions in harmonic analysis, geometric combinatorics, oscillatory integrals, dispersive and wave equations, and number theory have been uncovered. These applications typically involve quantitative estimates concerning Kakeya sets at a very small but positive resolution δ. This can equivalently be thought of as understanding configurations of families of overlapping *tubes* (i.e. thin cylinders with dimensions $\delta \times \cdots \times \delta \times 1$) which point in many different directions up to resolution δ.

Since a Kakeya set can have measure 0, the volume of a Kakeya set at resolution δ converges to 0 as δ converges to 0. One way to quantify this statement is to ask for the rate of convergence. If a set is still quite large, the volume should converge to 0 slowly. The *Kakeya conjecture* states that the volume should converge to 0 essentially as slowly as possible: more precisely, the conjecture asks if for all $\varepsilon > 0$ that the volume is lower bounded by a positive constant multiplied by δ^{ε}. In other words, the Kakeya sets are among the largest of sets with measure 0.

The case in dimension 2 was first resolved by Roy Davies in 1971 [Dav71] (Davies was a PhD student of Besicovitch in Cambridge in 1954). Despite the resolution in dimension 2, it has since turned out that the Kakeya conjecture in

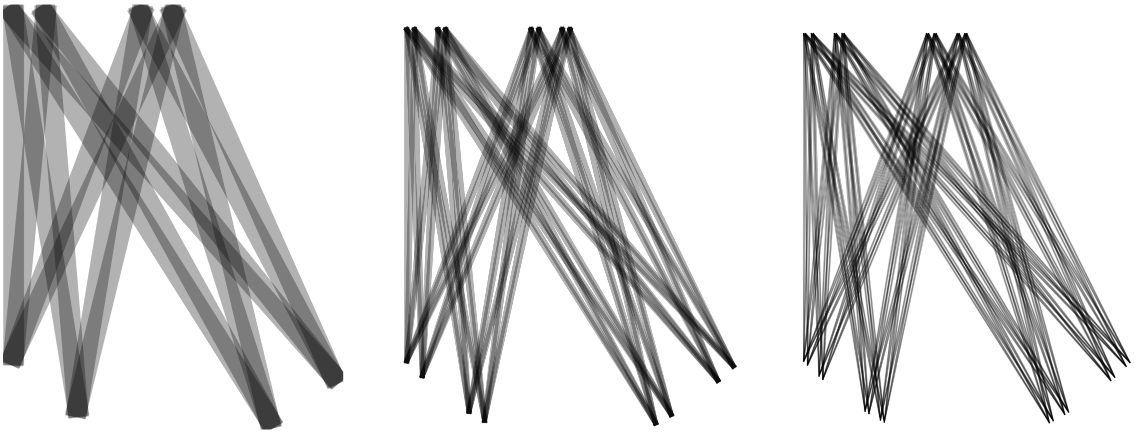

FIGURE 1. An approximation of a part of a Kakeya set at increasingly finer resolutions. The opacity of the region indicates the concentration of the tubes.

dimensions 3 and above is remarkably difficult and resisted resolution since then. However, at the end of February this year, Hong Wang (at New York University) and Joshua Zahl (at the University of British Columbia) announced the resolution of the Kakeya conjecture in dimension 3 [WZ25+]. This is the culmination of a multi-year effort of Wang and Zahl, and builds on a number of substantial breakthroughs in projection theory and geometric measure theory.

This conjecture has generated a large amount of interest over the years, and the resolution is considered by many to be one of the most substantial breakthroughs in geometric measure theory in its modern history. But first, to understand why such a question has generated so much interest, let us turn our attention to some of the history of problems closely related to the Kakeya set conjecture.

Convergence of Fourier integrals. In 1971, the same year as Davies' result on Kakeya sets in dimension 2, Charles Fefferman published a remarkable observation connecting tightly packed configurations of tubes and the Fourier transform [Fef71]. He gave a counterexample to a question known at the time as the *disc multiplier conjecture*. The disc multiplier conjecture involved the role of curvature in the convergence of the Fourier transform. Given a function $f \in L^p$ for some $1 \leq p \leq 2$, the *Fourier transform* of f is given by the integral

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x\cdot\xi} f(x)\,\mathrm{d}x.$$

The *Fourier inversion theorem* states, say for a smooth function f which decays quickly at infinity, that the original function f can be recovered from the Fourier transform:

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \widehat{f}(\xi)\,\mathrm{d}\xi.$$

While this can be made rigorous for more general functions in a distributional sense, again one is often interested in making more quantitative statements. The disc multiplier conjecture asked about the mode of convergence of the integral on the right hand side: namely, for L^p functions, do the bounded integrals

$$\int_{|x|\leq R} e^{2\pi i x\cdot\xi}\widehat{f}(\xi)\,\mathrm{d}\xi$$

converge in norm to f as R diverges to infinity? Plancherel's theorem indicates that this is indeed the case when $p = 2$. Remarkably, when $p \neq 2$, using a variant of Besicovitch's construction of a Kakeya set, Fefferman proved that such integrals need not converge. A key role is played by the *curvature* of the ball $B(0, R)$: replacing the ball by, say, a square $[-R, R]^d$, results in convergence in L^p for all $p \in (1, \infty)$.

A slightly less singular summation operator can be obtained by mollifying the indicator function on the ball $B(0, R)$ with the smooth cutoff function $(R - |\xi|)^\varepsilon$. Such operators are referred to as *Bochner–Riesz summation operators*. Since Besicovitch's construction of a Kakeya set only "barely" fails to have positive measure, one might hope that this slight modification is sufficient to guarantee convergence in L^p for a suitable open range of p. The *Bochner–Riesz conjecture* formulates the optimal range of values of p for which convergence can hold.

In the sense that a Kakeya set of measure 0 provides a counterexample for the sharp cutoff, quantifying the size of a Kakeya set is a pre-requisite for understanding the Bochner–Riesz conjecture. An influential intermediate formulation known as the *Kakeya maximal conjecture*, due to Antonio Córdoba in the mid 1970s, formulates more precise concentration estimate for sums of indicator functions on tubes [Cor77]. This is in contrast to the (comparatively) cruder volume estimates for unions of tubes. The Kakeya maximal conjecture is still open in $\mathbb{R}^3$.

Finally, another intermediate conjecture, which we only briefly mention here, is the *restriction conjecture for the sphere*, which seeks to understand when the Fourier transform of an L^p function (which is *a priori* only defined almost everywhere) can be naturally restricted to the surface of the sphere (which is a set of measure 0). Here again curvature is critical since analogous statements are not true, say, for hyperplanes. In the 1990s, Anthony Carbery showed that the restriction conjecture implies the Bochner–Riesz conjecture [Car92]; subsequently Terence Tao also showed the converse that the Bochner–Riesz conjecture also implies the restriction conjecture [Tao99].

Focusing properties of wave equations. Fefferman's counterexample to the disc multiplier conjecture exploited a general phenomenon concerning oscillatory integrals: even though the action of an operator may be difficult to analyze for general functions, the operator is quite well behaved on *wave packets*. The concept of a wave packet has its origins in the work of Erwin Schrödinger [Sch26] concerning the 1-dimensional harmonic oscillator. Even though Schrödinger originally hoped that wave packets would also remain spatially concentrated and provide a model for the electron, it was subsequently realized by Werner Heisenberg that such a wave packet cannot remain concentrated simultaneously in space and in frequency [Hei27]. This is the *Heisenberg uncertainty principle*.

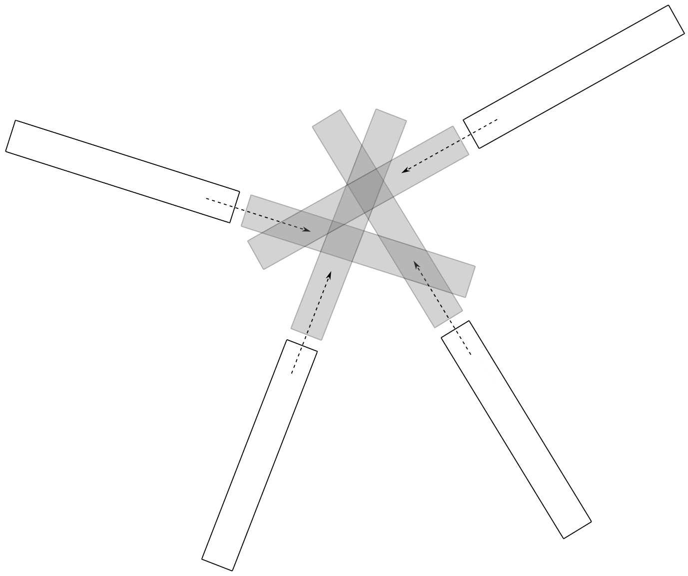

FIGURE 2. Wave packets in an initial non-concentrated configuration, and their translations.

From the perspective of modern analysis, a wave packet is a smooth function whose Fourier transform is essentially supported on a $R \times R^{1/2}$ rectangle (with any orientation). The fact that one chooses a decomposition into a highly eccentric rectangle, and not a square, is intimately related to the uncertainty principle. An important scheme introduced by Jean Bourgain in 1991 [Bou91] is to begin with a general function, decompose it into wave packets, apply the operator to each wave packet, and then reassemble the resulting wave packets. Here, Kakeya-type estimate play a critical role since, when R is very large, R is much larger than $R^{1/2}$ so the Fourier supports look like tubes. However, since the wave packets also contain oscillation, there is another substantial component to understand cancellation between different wave packets.

Wave packets in fact provide a more natural connection between Kakeya-type estimates and linear evolution equations, such as the *wave equation*:

$$\partial_{tt} u(t,x) = \Delta u(t,x); \quad u(0,x) = f(x); \quad \partial_t u(0,x) = 0.$$

A broad class of problems for such partial differential equations is to understand to what extent regularity assumptions on the initial data f imply regularity assumptions on the solutions at time t for $t > 0$.

For single time estimates, the situation is quite bad because of focusing issues. As a heuristic, one can imagine the problem of waves in a pool: even small waves can combine together to amplify to make abnormally tall waves at specific times (it is precisely "tall" and "spiky" functions which have large L^p norm for $p > 2$). However, if one is willing to average in time, one might expect a gain in regularity with the heuristic that waves cannot remain focused over extended time scales. Such estimates are known as *local smoothing*; a precise formulation for how much smoothing one might expect in the wave equation was given by Christopher Sogge in 1991 [Sog91]. The Kakeya conjecture is connected to local smoothing since the wave equation acts essentially by translation and dispersion on wave packets. Here, the fact that the wave packet decomposition is into tubes rather than balls means that even with dispersion the wave packet remains concentrated in approximately the same region as the initial region. Therefore a Kakeya set represents a configuration of wave packets which can begin un-concentrated, but then remain concentrated for a long period of time.

The local smoothing conjecture is the most general of all of the previously mentioned conjectures. Remarkably, even in two spatial dimensions (where we recall that the analogous volume bound for Kakeya sets has been known since 1970), it took until 2020 for the complete resolution of the local smoothing conjecture by Larry Guth, Hong Wang, and Ruixiang Zhang [GWZ20].

Since the Kakeya conjecture lies at the bottom of this collection of Fourier analytic conjectures, understanding the geometry of Kakeya sets is a critical prerequisite for making progress on other related questions. Such connections have provided much of the motivation behind Kakeya type problems.

Recent progress on the Kakeya conjecture. Let us now turn our attention to the resolution of the Kakeya conjecture in dimension 3, due to Wang and Zahl. Their proof builds on their earlier joint work, namely a key result concerning special Kakeya sets exhibiting a certain type of self-similarity [WZ22+], as well as a somewhat weaker estimate concerning the size of Kakeya sets [WZ24+].

The modern understanding of Kakeya sets in $\mathbb{R}^3$ perhaps begins with an argument of Thomas Wolff concerning "hairbrushes" (a "hairbrush" is a geometric configuration of tubes passing through a single "spine" tube, forming an object which looks like a hairbrush) [Wol95]. In particular, he established the lower bound of $\delta^{1/2}$ (up to δ^{ε} loss) for the volume of the δ-neighbourhood of Kakeya sets. However, Wolff's arguments in fact applied to a more general class of Kakeya-type sets, and within that class of sets there was a special configuration of lines (parametrized by the Heisenberg matrix group) for which his conclusions are optimal. This configuration of lines lives naturally in $\mathbb{C}^3$, rather than $\mathbb{R}^3$; so any improvement on Wolff's bounds would need to take into account the specific nature of Kakeya configurations, or special information concerning the geometry of the real line.

It turns out that a key feature here which distinguishes $\mathbb{R}$ from $\mathbb{C}$ is the presence of *large Borel sub-rings*. The complex plane has $\mathbb{R}$ as a Borel sub-ring (which is a 1 dimensional submanifold): but it turns out, due to a result of Edgar and Miller [EM03], and independently Bourgain [Bou03], that all proper Borel sub-rings of $\mathbb{R}$ must be very small (in an appropriate sense, zero dimensional). Crucially, Bourgain proved something much more quantitative, which is referred to as the *discretized sum-product phenomenon*: a subset of $\mathbb{R}$ at resolution δ cannot simultaneously have small growth under sumsets and under products. In 2014, Terence Tao proposed on his blog a possible approach (which he developed with Nets Katz) using Bourgain's discretized sum-product theory: his general scheme was to take a putative minimal counterexample to the Kakeya conjecture (i.e. a counterexample with volume scaling like δ^{σ} for some $\sigma > 0$ as large as possible), and then use the extremal nature of such a set to establish three properties which he called *stickiness* (that tubes want to cluster together in a roughly self-similar fashion), *graininess* (that tubes intersecting a given ball concentrate into rectangular slabs), and *planiness* (that the tubes passing through a common point are almost coplanar) [Tao14]. These properties could then be combined to obtain a contradiction with the sum-product theorem. The ideas for such an approach originated in Tao's earlier joint work with Nets Katz and Izabella Łaba from 2000 which improved on Wolff's conclusion with an exponent $1/2 - 10^{-10}$ [KŁT00].

The earlier work of Wang and Zahl resolved the "sticky" Kakeya conjecture: making the blanket assumption that the Kakeya set satisfied a suitable analogue of stickiness, they then went on to prove the conjectured lower bound [WZ22+]. Another pillar of the work of Wang and Zahl is an adaptation of an argument due to Larry Guth from 2016 which states that any putative counterexample to the Kakeya conjecture must satisfy a precise type of graininess [Gut16]. Finally, planiness is closely related to a *multilinear* variant of the Kakeya conjecture due to Jonathan Bennett, Anthony Carbery, and Terence Tao from 2006 [BCT06]. The results concerning sticky Kakeya sets also critically involved recent progress in projection theory and additive combinatorics [DZ16; OSW24; PYZ22+], which in turn have their foundations in the discretized sum-product theory of Bourgain.

In practice, assembling the myraid of components to complete the proof of the full conjecture requires many new ideas. In broad lines, the key new insight of Wang and Zahl is an induction scheme using the decomposition into grains. By contradiction, they begin with an extremal counterexample to the Kakeya conjecture, and then obtain a decomposition into maximal grains. Then the argument converts into casework concerning the configurations of the grains: either the dimension of the corresponding set must be larger than expected (for instance if it falls into one of the previously established cases of the Kakeya conjecture), or one can locate a new resolution and improve the configuration, and repeat. This method of either gaining an improvement, or locating a new resolution, is often referred to as *induction on scale*.

The details of this scheme are given in a 127 page preprint, currently posted on the arXiv preprint server [WZ25+]. While the proof is not yet published, it is broadly believed that the results are correct. The resolution of the Kakeya conjecture in $\mathbb{R}^3$ is the first step towards understanding many questions and conjectures building on Kakeya-type estimates, and contributes to our understanding of the fundamental nature of the Fourier transform and the connections to other areas of mathematics and physics.

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