On L^q-spectra of Self-similar Measures

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ABSTRACT. We investiate and discuss basic properties of the L^q-spectrum of self-similar measures in the real line.

Contents

1. INTRODUCTION AND BASIC RESULTS

1.1. L^q**-spectra.** Let μ be a compactly-suppored Borel probability measure in $\mathbb{R}$. Given $q \in \mathbb{R}$ and $r > 0$, define

$$\Theta(\mu, q; r) := \sup\bigl\{\sum_i \mu\bigl(B(x_i, r)\bigr)^q : \{B(x_i, r)\}_i \text{ is a centred packing of } \operatorname{supp}\mu]\bigr\}.$$

Then the L^q*-spectrum* of μ is the map

$$q \mapsto \tau(\mu, q) := \liminf_{r\to 0} \frac{\log \Theta(\mu, q; r)}{\log r}.$$

Proposition 1.1. *Let μ be a Borel probability measure. Then $\tau(\mu, q)$ is a nondecreasing concave function of q.*

Proof. Let $q_1 < q_2$ be arbitrariy. That $\tau(\mu, q)$ is non-decreasing follows since $\mu(B(x_i, r))^{q_1} \geq \mu(B(x_i, r))^{q_2}$ for any $x_i \in K$.

In addition, concavity is a standard application of Hölder's inequality: let $0 < \lambda < 1$, and then with Hölder's inequality applied to $1/\lambda$ and $1/(1-\lambda)$, we have

$$\sum_i \mu\bigl(B(x_i, r)\bigr)^{\lambda q_1 + (1-\lambda) q_2} \leq \left(\sum_i \mu\bigl(B(x_i, r)\bigr)^{q_1}\right)^{\lambda} \left(\sum_i \mu\bigl(B(x_i, r)\bigr)^{q_2}\right)^{1-\lambda} \tag{1.1}$$

and taking suprema and logarithms yields concavity. □

When $q \geq 0$, the stability of taking positive powers means that the L^q-spectrum can be calculated as a sum over half-closed dyadic intervals. Given some $k \in \mathbb{N}$, let $\mathcal{D}_m = \{[j/2^k, (j+1)/2^k) : j \in \mathbb{Z}\}$. For $q \geq 0$ $S_m(\mu, q) = \sum_{J \in \mathcal{D}_m} \mu(J)^q$ (where we implicitly sum over J with $\mu(J) > 0$), and we have the following result:

2010 *Mathematics Subject Classification.* 28A80,37C45.
Key words and phrases. REPLACE.
This paper is in final form and no version of it will be submitted for publication elsewhere.

Proposition 1.2. *Let μ be a compactly supported Borel probability measure. Then*
$$\tau(\mu,q)=\liminf_{m\to\infty}\frac{-\log_2 S_m(\mu,q)}{m}.$$

Proof. We adopt this proof from [?]. Let $\{B(x_i,r)\}_i$ be a centred packing of $\operatorname{supp}\mu$. Let m be such that $2^{-m-1}<r\leq 2^{-m}$ so that each $B(x_i,r)$ can be covered by at most 2 elements of $\mathcal{D}_m$, denoted by $\mathcal{J}_i$. Then by Jensen's inequality for the convex function x^q, we have
$$\begin{aligned}\sum_i \mu\big(B(x_i,r)\big)^q &\leq \sum_i \left(\sum_{J\in\mathcal{J}_i}\mu(J)\right)^q \leq \sum_i 2^{q-1}\sum_{J\in\mathcal{J}_i}\mu(J)^q\\ &\leq 2^q\sum_{J\in\mathcal{D}_m}\mu(J)^q\end{aligned}$$
from which it follows that
$$\tau(\mu,q)\geq\liminf_{m\to\infty}\frac{-\log_2 S_m(\mu,q)}{m}.$$

Conversely, given any $m>0$, enumerate $\{J\in\mathcal{D}_m:\mu(J)>0\}=\{J_i\}_{i=1}^M$, and for each i, get $x_i\in J_i\cap(\operatorname{supp}\mu)$, so that $B(x_j,2^{-m})\supseteq J$. Note that for any $j_1<j_2<j_3$, we have $B(x_{j_1},2^{-m})\cap B(x_{j_2},2^{-m})\cap B(x_{j_3},2^{-m})=\emptyset$. Thus there exists a subcollection $\{B(x_{j_i},2^{-m})\}_{i=1}^k$ such that
$$\sum_{i=1}^k\mu\big(B(x_{j_i})\big)^q\geq\frac{1}{3}\sum_{i=1}^M\mu\big(B(x_j,2^{-m})\big)^q\geq\frac{1}{3}\sum_{J\in\mathcal{D}_m}\mu(J)^q.$$
(Note that the greedy choice guarantees that such a collection exists: having chosen $B(x_{j_1},2^{-m}),\dots,B(x_{j_i},2^{-m})$, choose j_{i+1} to be the index in $\{1,\dots,M\}$ with $B(x_{j_{i+1}},2^{-m})$ disjoint from the previously chosen balls with maximal measure.) We thus have
$$\Theta(\mu,q;2^{-m})\geq\frac{1}{3}S_m(\mu,q)$$
so that $\tau(\mu,q)\leq\liminf_{m\to\infty}\frac{-\log_2 S_m(\mu,q)}{m}$ and equality holds, as claimed.

□

Remark 1.3. *Note that this proof only works for $q\geq 0$ since the sums $\Theta(\mu,q;r)$ and $S_m(\mu;q)$ are governed by the elements with largest mass. For $q<0$, one may encounter issues where some dyadic interval $J\in\mathcal{D}_m$, which need not be centred in $\operatorname{supp}\mu$, intersects $\operatorname{supp}\mu$ on a set with disproportionately small measure. It can be shown that replacing the family $\mathcal{D}_m$ by the family $\{[(j-1)/2^m,(j+2)/2^m):j\in\mathbb{Z}\}$ (as used by Riedi* [?]*) one has an equivalent definition, but we do not include a proof here.*

Using this equivalent formulation of the L^q-spectrum for $q\geq 0$ in terms of dyadic intervals, we have the following observation:

Corollary 1.4. *We have $\tau(\mu,0)=-\underline{\dim}_B\operatorname{supp}\mu$ and $\tau(\mu,1)=0$.*

1.2. **L^q-dimensions.** For any $q > 1$, we define the *L^q-dimension* of μ by

$$D(\mu, q) = \frac{\tau(\mu, q)}{q - 1}.$$

As we will see, the normalization factor $1/(q-1)$ is chosen since it is the ratio between q and its conjugate exponent. We can make some basic observations about L^q-dimensions:

Proposition 1.5. *Let μ be a compactly supported Borel probability measure.*

(i) $D(\mu, q)$ is non-decreasing.
(ii) We have $D(\mu, q) \in [0, 1]$.
(iii) If μ is purely atomic, then $D(\mu, q) = 0$ for any $q > 1$.
(iv) If μ is absolutely continuous with respect to Lebesgue measure with density in $\mathcal{L}^q$ for some $q > 1$, then $D(\mu, q) = 1$.

Proof. For convenience, we rescale so that $\operatorname{supp}\mu \subseteq [0, 1]$.

(i) Arguing similarly to **??** in **??**, with $q_1 < q_2$ and $0 < \lambda < 1$, we observe that $S_m(\mu; q_1 + (1-\lambda)q_2) \leq S_m(\mu; q_1)^\lambda S_m(\mu; q_2, r)^{1-\lambda}$. Now given $1 < p < q$, take $p_1 = 1$, $p_2 = q$, and $\lambda = (p-1)/(q-1)$ so that $\lambda p_1 + (1-\lambda)p_2 = p$ and
$$S_m(\mu, p) \leq S_m(\mu, 1)^\lambda S_m(\mu, q)^{1-\lambda} = S_m(\mu, q)^{1-\lambda} \leq S_m(\mu, q)$$
since $S_m(\mu, 1) = 1$ and $S_m(\mu, q) \leq 1$, from which the conclusion follows.

(ii) We always have $S_m(\mu, q) \leq 1$ so that $D(\mu, q) \geq 0$. Then by Hölder's inequality, we have
$$1 = \sum_{J \in \mathcal{D}_m} 1 \cdot \mu(J) \leq (2^m)^{(q-1)/q} S_m(\mu, q)^{1/q},$$
that is $2^{m(1-q)} \leq S_m(\mu, q)$ and $D(\mu, q) \leq 1$.

(iii) If μ is purely atomic, then for m sufficiently large, $S_m(q)$ is constant.

(iv) If μ has density $f \in \mathcal{L}^q$, then for any $m \in \mathbb{N}$ and $J \in \mathcal{D}_m$, we have by Hölder's inequality
$$\mu(J) \leq \left(\int_J f^q \,\mathrm{d}m \right)^{1/q}$$
and thus $S_m(\mu, q) \leq \int_0^1 f^q \,\mathrm{d}m \leq M < \infty$ [***TODO: not quite sure how this works***]

□

1.3. **Frostman exponents.** Frostman exponents are closely related to the L^q-spectrum of the measure μ. We say that the measure μ has *Frostman exponent* s if there exists some $C > 0$ such that
$$\mu\big(B(x, r)\big) \leq C r^s$$
for all $x \in \operatorname{supp}\mu$ and $r > 0$. We emphasize here that the this inequality must hold everywhere, and not simply μ-almost everywhere. We then define
$$\dim_\infty \mu := \sup\{t \geq 0 : \mu \text{ has Frostman exponent } t\}.$$

It is clear that

$$t \leq \inf_{x \in \operatorname{supp} \mu} \underline{\dim}_{\mathrm{loc}}(\mu, x) \leq \overline{\dim}_B \operatorname{supp} \mu.$$

The relationship between Frostman exponents and L^q-spectra is given by the following result. The proof is due to Fraser and Jordan [**?**, Lem. 2.1]:

Proposition 1.6 ([**?**])**.** *Let μ be a compactly supported Borel probability measure. Then*

$$\dim_\infty \mu = \inf\{t \geq 0 : \tau(q) < tq \text{ for all } q \geq 0\}.$$

Proof. Suppose $t, q > 0$ have $\tau(\mu, q) = tq$, and let $\epsilon > 0$ be arbitrary. Then for any $r > 0$ sufficiently small, we have

$$\Theta(\mu, q; r) < r^{(t-\epsilon)q}.$$

Thus there exists $C > 0$ such that, if $x \in \operatorname{supp} \mu$ is arbitrary, we have

$$\mu\big(B(x, r)\big)^q \leq \Theta(\mu, q; r) < Cr^{(t-\epsilon)q}$$

so that $\dim_\infty \mu \geq t - \epsilon$. But $\epsilon > 0$ was arbitrary so that

$$\dim_\infty \mu \geq \inf\{t \geq 0 : \tau(q) < tq \text{ for all } q \geq 0\}.$$

Conversely, let $0 \leq t < \dim_\infty \mu$ be arbitrary so there exists some $C > 0$ such that $\mu(B(x, r)) \leq Cr^t$ for any $r > 0$ and $x \in \operatorname{supp} \mu$. Rescaling μ if necessary, we may assume $B(x, r) \subseteq [0, 1]$ for all $x \in \operatorname{supp} \mu$ and $0 < r < 1$. Let $\mathcal{B} = \{B(x_i, r)\}_i$ be a centred packing of $\operatorname{supp} \mu$. Then $\#\mathcal{B} \leq r^{-1}$ so that

$$\sum_i \mu\big(B(x_i, r)\big)^q \leq Cr^{tq-1}$$

and thus $\Theta(\mu, q; r) \leq Cr^{tq-1}$ and

$$\frac{\tau(\mu, q)}{q} \geq t - \frac{1}{q}.$$

But if $t_1 < t$ is arbitrary, then there is $q > 0$ such that $\tau(\mu, q)/q > t_1$ and thus

$$\dim_\infty \mu \leq \inf\{t \geq 0 : \tau(q) < tq \text{ for all } q \geq 0\}$$

as claimed. □

Remark 1.7. *In other words, the Frostman exponent is the slope of the asymptote of $\tau(\mu, q)$ as q tends to infinity.*

2. Basic properties of L^q-spectra

Proposition 2.1. *Let μ be a compactly supported Borel probability measure with L^q-spectrum $\tau(\mu, q)$ and fine multifractal spectrum*

$$f(\mu, \alpha) := \dim_H\{x \in \operatorname{supp} \mu : \overline{\dim}_{\mathrm{loc}}(\mu, x) = \underline{\dim}_{\mathrm{loc}}\mu(x) = \alpha\}.$$

Then $f(\mu, \alpha) \leq \tau^*(\mu, \alpha)$.

Proof. Translate from [**?**, Thm. 4.1]. □

2.1. **Concave functions.**

3. L^q-SPECTRA OF SELF-SIMILAR MEASURES

3.1. **Self-similar measures.** Let $\mathcal{I}$ be a non-empty finite set of indices. We then say that $\{S_i\}_{i\in\mathcal{I}}$ with each $S_i : \mathbb{R} \to \mathbb{R}$ is an *iterated function system of similarities* (IFS) if

$$S_i(x) = r_i x + d_i \text{ where } 0 < |r_i| < 1 \text{ for each } i \in \mathcal{I}.$$

To any IFS there exists a unique compact set $K \subseteq \mathbb{R}$ such that

$$K = \bigcup_{i\in\mathcal{I}} S_i(K).$$

We call K a *self-similar set*. Moreover, if $(p_i)_{i\in\mathcal{I}}$ is a probability vector, there exists a unique Borel measure $\mu_{\boldsymbol{p}}$ satisfying

$$\mu_{\boldsymbol{p}}(E) = \sum_{i\in\mathcal{I}} p_i \mu_{\boldsymbol{p}} \circ S_i^{-1}(E).$$

We refer to $\mu_{\boldsymbol{p}}$ as a *self-similar measure* For non-degeneracy, we will assume throughout that the K is not a singleton and $p_i > 0$ for all $i \in \mathcal{I}$. In this case, we always have $\operatorname{supp}\mu_{\boldsymbol{p}} = K$. Note that both K and $\mu_{\boldsymbol{p}}$ can be realized as the fixed point of an appropriate contracting map on some compact metric space and uniqueness follows; see **[?]** for details.

For each $n \in \mathbb{N}$, $\mathcal{I}^n$ is the set of words of length n and $\mathcal{I}^* = \bigcup_{i=0}^{\infty} \mathcal{I}^n$ is the set of all finite words. Given $\sigma = (i_1, \ldots, i_n) \in \mathcal{I}^*$, we write $p_\sigma = p_{i_1} \cdots p_{i_n}$, $r_\sigma = r_{i_1} \cdots r_{i_n}$ and $S_\sigma = S_{i_1} \circ \cdots \circ S_{i_n}$. When $n \geq 1$, we write $\sigma^- = (i_1, \ldots, i_{n-1})$. If $\emptyset$ denotes the empty word, then $S_\emptyset$ is the identity map and $p_\emptyset = r_\emptyset = 1$. We denote by $[\sigma] = \{(x_j)_{j=1}^\infty \in \mathcal{I}^{\mathbb{N}} : x_j = i_j \text{ for each } 1 \leq j \leq n\}$.

Given some $r > 0$ and a Borel set $E \subseteq \mathbb{R}$, define

$$\begin{aligned} \Lambda_r &= \{\sigma \in \mathcal{I}^* : |r_\sigma| < r \leq |r_{\sigma^-}|\} \\ \Lambda_r(E) &= \{\sigma \in \Lambda_r : S_\sigma(K) \cap E \neq \emptyset\}. \end{aligned}$$

The following result follows directly from the definitions:

Lemma 3.1. *Let $r > 0$. Then*

(i) $\{[\sigma] : \sigma \in \Lambda_r\}$ *is a partition of* $\mathcal{I}^{\mathbb{N}}$.
(ii) $K = \bigcup_{\sigma\in\Lambda_r} S_\sigma(K)$ *and* $\mu_{\boldsymbol{p}} = \sum_{\sigma\in\Lambda_r} p_\sigma \mu_{\boldsymbol{p}} \circ S_\sigma^{-1}$.
(iii) $\mu_{\boldsymbol{p}}(E) = \sum_{\sigma\in\Lambda_r(E)} p_\sigma \mu_{\boldsymbol{p}} \circ S_\sigma^{-1}(E)$.

Proof. Note that (i) follows from the observation that for any $r > 0$, each sequence $\overline{\sigma} \in \mathcal{I}^n$ has a unique prefix in Λ_r. Part (ii) follows by iterating the similarity properties of K and $\mu_{\boldsymbol{p}}$, and part (iii) follows from (ii) by removing elements of the sum for which $S_\sigma^{-1}(E) \cap K = \emptyset$. □

3.2. **Regularity and points of differentiability.**

Proposition 3.2. *Let μ be a Borel probability measure in $\mathbb{R}^d$. Suppose $\tau(\mu, q)$ is differentiable at $q = 1$. Then $\tau'(\mu, 1) = \dim_H \operatorname{supp}(\mu_{\boldsymbol{p}})$.*

Proof. [**?**] □

Proposition 3.3. *Let μ be a self-similar measure. Then $\dim_H \operatorname{supp} \mu = \lim_{q \to 1^+} D(\mu, q)$.*

Proof. [**?**, Thm. 5.1 and Rem. 5.2] □

Theorem 3.4. *Let μ be a self-similar measure in $\mathbb{R}^d$. Suppose $\tau(\mu, q)$ is differentiable at $q \geq 1$. Then the $\dim_H K(\mu, \alpha) = \tau^*(\alpha)$.*

Proof. [**?**, Thm. 1.1]. □

3.3. **Existence of limit for $q \geq 0$.**

3.4. **Frostman exponents.**

3.5. **Exact-dimensionality.**

3.6. **Proving Differentiability.** Let μ be a compactly supported Borel probability measure. Recall that $\mathcal{D}_n$ is the partition of $\mathbb{R}$ into the Dyadic intervals $[k/2^n, (k+1)/2^n)$ for integer k. For convenience, we code the elements of $\mathcal{D}_n$ in the form $I_{j_1 \dots j_n}$ such that $I_{j_1 \dots j_n} \in \mathcal{D}_n$ and $I_{j_1 \dots j_n} \supset I_{j_1 \dots j_n}$. If $I = I_{j_1 \dots j_n} \in \mathcal{D}_n$ and $J = I_{j_{n+1} \dots j_{n+m}} \in \mathcal{D}_m$, we denote

$$IJ = I_{j_1 \cdots j_{n+m}} \in \mathcal{D}_{n+m}.$$

Proposition 3.5. *Let μ be as above, and suppose there exists some $C > 0$ such that for any $I \in \mathcal{D}_n$ and $J \in \mathcal{D}_m$, we have $\mu(IJ) \leq C\mu(I)\mu(J)$.*

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