Monotonization of dimension spectra

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ABSTRACT. The goal of this note is to provide simple proofs of the following results. Let X be a non-empty metric space.

(i) For all $0 < \theta < 1$,

$$\underline{\dim}_{\mathrm{L}}^{\theta} X = \inf_{0<\lambda\leq\theta} \dim_{\mathrm{L}}^{\lambda} X.$$

(ii) Suppose in addition that there is a number $s < \infty$ and a constant $C > 0$ such that $N_r(B(x,1)) \leq Cr^{-s}$ for all $0 < r \leq 1$. Then for all $0 < \theta < 1$,

$$\overline{\dim}_{\mathrm{A}}^{\theta} X = \inf_{0<\lambda\leq\theta} \dim_{\mathrm{A}}^{\lambda} X.$$

Monotonization of the lower spectrum. Let X be a non-empty metric space and let γ be defined for $v \leq u$ by the rule

$$\gamma(u,v) = \log \inf_{x\in K} N_{2^{-u}}(B(x, 2^{-v})).$$

Here, the logarithm is in base 2. Note that γ takes values in $[0,\infty]$, since we make no assumptions about the metric space X. It is easy to see that $v \mapsto \gamma(u,v)$ is a decreasing function for each fixed u.

We recall the *lower spectrum* of X defined for $0 < \theta < 1$ and denoted by $\dim_{\mathrm{L}}^{\theta} X$: it is the supremum over all $s \geq 0$ for which there exists an $m_s \geq 0$ such that for all $u \geq 0$

$$\gamma(u, \theta u) \geq su(1-\theta) - m_s. \tag{0.1}$$

We also recall the *monotone lower spectrum* of X defined for $0 < \theta < 1$ and denoted by $\underline{\dim}_{\mathrm{L}}^{\theta} X$: it is the supremum over all $s \geq 0$ for which there exists an $m_s \geq 0$ such that for all $u \geq 0$ and $0 \leq \lambda \leq \theta$,

$$\gamma(u, \lambda u) \geq su(1-\lambda) - m_s.$$

The following result was proven under the additional assumption that X is doubling and uniformly perfect in [CWC20, Theorem 1.1], and then with only the doubling assumption in [HT21, Theorem A.2]. Here we dispense of the doubling assumption, and moreover give a simple proof which only uses monotonicity of γ.

Theorem 0.1 ([BCRW26+]). *Let X be a non-empty metric space. Then for all $0 < \theta < 1$,*

$$\underline{\dim}_{\mathrm{L}}^{\theta} X = \inf_{0<\lambda\leq\theta} \dim_{\mathrm{L}}^{\lambda} X.$$

We first note a standard equivalent formula for the lower spectrum.

Lemma 0.2. *Let X be a non-empty metric space and $\theta \in (0,1)$. Then*

$$\dim_{\mathrm{L}}^{\theta} X = \liminf_{u\to\infty} \frac{\gamma(u,\theta u)}{u(1-\theta)}.$$

Proof. Write $\Lambda = \liminf_{u\to\infty} \frac{\gamma(u,\theta u)}{u(1-\theta)}$. On one hand, if $s \geq 0$ and $m_s \geq 0$ satisfy (0.1), dividing by u and taking a limit infimum, it is immediate that $\Lambda \geq s$. On the other hand, suppose t is such that for all $m \geq 0$ there is a $u_m \geq 0$ so that

$$\gamma(u_m, \theta u_m) \leq t(1-\theta)u_m - m.$$

Since γ is non-negative, we must have $\lim_{m\to\infty} u_m = \infty$ and therefore $\Lambda(\gamma) \leq t$. We conclude that $(1-\theta)\dim_{\mathrm{L}}^{\theta} K = \Lambda$. □

Proof (of Theorem 0.1). It suffices to prove that

$$\underline{\dim}_{\mathrm{L}}^{\theta} X \geq \inf_{0<\lambda\leq\theta} \dim_{\mathrm{L}}^{\lambda} X \tag{0.2}$$

since the opposite inequality is immediate from the definition.

We may moreover assume that $\underline{\dim}_{\mathrm{L}}^{\theta} X < \infty$, or else the inequality is trivial. Let $t > s > \underline{\dim}_{\mathrm{L}}^{\theta} X$ be arbitrary. By definition of $\underline{\dim}_{\mathrm{L}}^{\theta} X$, for all $n \in \mathbb{N}$ there is a $u_n \geq 0$ and $\lambda_n \in [0,\theta]$ such that

$$\gamma(u_n, \lambda_n u_n) \leq s u_n(1-\lambda_n) - n.$$

Since γ is non-negative, we must have $\lim_{n\to\infty} u_n = \infty$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} \lambda_n = \lambda \in [0,\theta]$. If $\lambda = \theta$, then $\lambda_n \leq \theta$ for all $n \in \mathbb{N}$ so that

$$\gamma(u_n, \theta u_n) \leq \gamma(u_n, \lambda_n u_n) \leq s u_n(1-\lambda_n).$$

Since $\theta < 1$, $\lim_{n\to\infty}(1-\lambda_n)^{-1} = (1-\theta)^{-1}$ and therefore by Lemma 0.2,

$$\dim_{\mathrm{L}}^{\theta} X \leq \liminf_{n\to\infty} \frac{\gamma(u_n, \theta u_n)}{u_n(1-\lambda_n)} \leq s < t.$$

Otherwise, $\lambda < \theta$, and let $\lambda < \kappa \leq \theta$ be such that $s(1-\lambda)/(1-\kappa) < t$. Since $\lambda_n \leq \kappa$ for all n sufficiently large,

$$\gamma(u_n, \kappa u_n) \leq \gamma(u_n, \lambda_n u_n) \leq s u_n(1-\lambda_n).$$

and therefore, exactly as before,

$$\dim_{\mathrm{L}}^{\kappa} X \leq \liminf_{n\to\infty} \frac{\gamma(u_n, \lambda_n u_n)}{u_n(1-\lambda_n)} \cdot \frac{1-\lambda_n}{1-\kappa} \leq s\frac{1-\lambda}{1-\kappa} \leq t.$$

In any case, we have shown for all $t > \underline{\dim}_{\mathrm{L}}^{\theta} X$ there is a $\lambda \in (0,\theta]$ so that $\dim_{\mathrm{L}}^{\lambda} X \leq t$. Therefore (0.2) follows. □

Monotonization of the Assouad spectrum. Similarly to before, we define

$$\beta(u,v) = \log \sup_{x\in K} N_{2^{-u}}(B(x,2^{-v})).$$

Here, the logarithm is in base 2. Note that β takes values in $[0,\infty]$, since we make no assumptions about the metric space X. It is easy to see that $v \mapsto \beta(u,v)$ is a decreasing function for each fixed u. Moreover, β is subadditive: for all $v \le w \le u$,

$$\beta(u,v) \le \beta(u,w) + \beta(w,v).$$

We recall the *Assouad spectrum* of X defined for $0 \le \theta < 1$ and denoted by $\dim_{\mathrm{L}}^{\theta} X$: it is the infimum over all $s \ge 0$ for which there exists an $m_s \ge 0$ such that for all $u \ge 0$

$$\beta(u,\theta u) \le su(1-\theta) + m_s. \tag{0.3}$$

We also recall the *monotone Assouad spectrum* of X defined for $0 \le \theta < 1$ and denoted by $\underline{\dim}_{\mathrm{L}}^{\theta} X$: it is the infimum over all $s \ge 0$ for which there exists an $m_s \ge 0$ such that for all $u \ge 0$ and $0 \le \lambda \le \theta$,

$$\beta(u,\lambda u) \le su(1-\lambda) + m_s.$$

For $\theta = 0$, we obtain a local variant of the upper box dimension with constants which are uniform over all balls.

Lemma 0.3. *Let X be a non-empty metric space and suppose $\dim_{\mathrm{A}}^{0} X < \infty$. Then for all $0 < \theta < 1$,*

$$\dim_{\mathrm{A}}^{0} X \le \dim_{\mathrm{A}}^{\theta} X \le \overline{\dim}_{\mathrm{A}}^{\theta} X \le \frac{\dim_{\mathrm{A}}^{0} X}{1-\theta}.$$

Proof. We begin with the first inequality. If $\dim_{\mathrm{A}}^{0} X = 0$ there is nothing to prove. Otherwise, let $0 < \varepsilon < \dim_{\mathrm{A}}^{0} X$ be arbitrary. Then there is a constant $m_s \ge 0$ and a sequence u_n with $\lim_{n\to\infty} u_n = \infty$ so that

$$\begin{aligned}(\dim_{\mathrm{A}}^{0} X - \varepsilon)\cdot u_n - m_s &\le \beta(u_n, 0)\\ &\le \beta(u_n,\theta u_n) + \beta(\theta u_n, 0)\\ &\le (\dim_{\mathrm{A}}^{0} X + \varepsilon)\theta u_n + (\dim_{\mathrm{A}}^{\theta} X + \varepsilon)u_n(1-\theta) + m_s.\end{aligned}$$

Dividing by u_n and taking the limit in n,

$$(1-\theta)\dim_{\mathrm{A}}^{0} X \le (1-\theta)\dim_{\mathrm{A}}^{\theta} X + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $\dim_{\mathrm{A}}^{0} X \le \dim_{\mathrm{A}}^{\theta} X$.

Now, the second inequality is trivial by definition. For the final inequality, let $\varepsilon > 0$ be arbitrary. Then there is an $m_s \ge 0$ so that for all $0 \le \lambda \le \theta$ and $u \ge 0$,

$$\beta(u,\lambda u) \le \beta(u,0) \le (\dim_{\mathrm{A}}^{0} X + \varepsilon)u + m_s \le \frac{(\dim_{\mathrm{A}}^{0} X + \varepsilon)}{1-\theta}\cdot u(1-\theta) + m_s.$$

Therefore $\overline{\dim}_{\mathrm{A}}^{\theta} X \le (\dim_{\mathrm{A}}^{0} X + \varepsilon)/(1-\theta)$, and since $\varepsilon > 0$ was arbitrary the proof is complete. □

As before, we also note the following lemma.

Lemma 0.4. *Let X be a non-empty metric space with $\dim_{\mathrm{A}}^0 X < \infty$. Then for all $\theta \in (0,1)$,*

$$\dim_{\mathrm{A}}^\theta X = \limsup_{u\to\infty} \frac{\beta(u,\theta u)}{u(1-\theta)}.$$

Proof. Write $\Lambda = \limsup_{u\to\infty} \frac{\beta(u,\theta u)}{u(1-\theta)}$. On one hand, if $s \geq 0$ and $m_s \geq 0$ satisfy (0.3), dividing by u and taking a limit infimum, it is immediate that $\Lambda \leq s$.

On the other hand, suppose t is such that for all $m \geq 0$ there is a $u_m \geq 0$ so that

$$\beta(u_m, \theta u_m) \geq t(1-\theta)u_m + m.$$

Since $\dim_{\mathrm{A}}^0 X < \infty$, there are a constants $\alpha, k \geq 0$ so that $\beta(u,\theta u) \leq \alpha u + k$ for all $u \geq 0$. Thus we must have $\lim_{m\to\infty} u_m = \infty$ and therefore $\Lambda \geq t$. We conclude that $(1-\theta)\dim_{\mathrm{L}}^\theta K = \Lambda$. □

We can now prove our main result concerning the Assouad spectrum. This was proven in [FHH+19] under the additional assumption that X is doubling, and in [OR25+] under the slightly weaker assumption that $\lim_{\theta\to 1} \overline{\dim}_{\mathrm{A}}^\theta X < \infty$.

Theorem 0.5. *Let X be a non-empty metric space with $\dim_{\mathrm{A}}^0 X < \infty$. Then for all $0 < \theta < 1$,*

$$\overline{\dim}_{\mathrm{A}}^\theta X = \sup_{0<\lambda\leq\theta} \dim_{\mathrm{L}}^\lambda X.$$

Proof. It suffices to prove that

$$\overline{\dim}_{\mathrm{A}}^\theta X \leq \sup_{0<\lambda\leq\theta} \dim_{\mathrm{A}}^\lambda X \tag{0.4}$$

since the opposite inequality is immediate from the definition.

If $\overline{\dim}_{\mathrm{A}}^\theta X = 0$ there is nothing to prove. Otherwise, let $0 \leq t < s < \underline{\dim}_{\mathrm{L}}^\theta X$ be arbitrary. By definition of $\overline{\dim}_{\mathrm{A}}^\theta X$, for all $n \in \mathbb{N}$ there is a $u_n \geq 0$ and $\lambda_n \in [0,\theta]$ such that

$$\beta(u_n, \lambda_n u_n) \geq s u_n(1-\lambda_n) + n.$$

As noted in the proof of Lemma 0.4, we must have $\lim_{n\to\infty} u_n = \infty$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} \lambda_n = \lambda \in [0,\theta]$. If $\lambda = 0$, then $\lambda_n \geq 0$ for all $n \in \mathbb{N}$ so that

$$\beta(u_n, 0) \geq \beta(u_n, \lambda_n u_n) \geq s u_n(1-\lambda_n).$$

Then $\lim_{n\to\infty}(1-\lambda_n)^{-1} = 1$ and therefore by Lemma 0.4,

$$\dim_{\mathrm{A}}^\theta X \geq \dim_{\mathrm{A}}^0 X \geq \liminf_{n\to\infty} \frac{\beta(u_n, \theta u_n)}{u_n(1-\lambda_n)} \geq s > t.$$

Otherwise, $\lambda > 0$, and let $0 < \kappa < \lambda$ be such that $s(1-\lambda)/(1-\kappa) > t$. Since $\lambda_n \geq \kappa$ for all n sufficiently large,

$$\beta(u_n, \kappa u_n) \geq \beta(u_n, \lambda_n u_n) \geq s u_n(1-\lambda_n).$$

and therefore, exactly as before,

$$\dim_{\mathrm{L}}^{\kappa} X \leq \limsup_{n\to\infty} \frac{\beta(u_n, \lambda_n u_n)}{u_n(1-\lambda_n)} \cdot \frac{1-\lambda_n}{1-\kappa} \geq s\frac{1-\lambda}{1-\kappa} \geq t.$$

In any case, we have shown for all $t < \overline{\dim}_{\mathrm{A}}^{\theta} X$ there is a $\lambda \in (0, \theta]$ so that $\dim_{\mathrm{A}}^{\lambda} X \geq t$. Therefore (0.2) follows. □

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