Convexity and Subadditivity

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ABSTRACT. This work in progress discusses various properties of functions which satisfy some form of convexity or subadditivity, with a focus on functions satisfying both.

CONTENTS

1. SUBADDITIVITY

Definition 1.1. *Let A be an abelian semigroup (typically $\mathbb{R}$, $\mathbb{N}$, $\mathbb{R}^d$, etc.). We say that $f\colon A \to \mathbb{R}$ is* subadditive *if*

$$f(x+y) \leq f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

A natural first example of a subadditive function is a sequence $(a_n)_{n=1}^{\infty} \subset \mathbb{R}$ satisfying $a_{n+m} \leq a_n + a_m$. As a fundamental illustration of the nice properties of subadditivity, we have the following result due to Fekete [1]:

Lemma 1.2 (Subadditivity). *If $(a_i)_{i=1}^{\infty}$ is subadditive, then $\lim_{n\to\infty} a_n/n$ exists and is equal to its infimum $L \coloneqq \inf_{n\geq 1} a_n/n$.*

Proof. For any $\epsilon > 0$, let n be such that $a_n/n < L + \epsilon$ and $b = \max\{a_i : 1 \leq i \leq n\}$. For $m \geq n$, write $m = qn + r$ with $0 \leq r < n$. Then from the subadditivity property, we have

$$a_{qn+r} \leq qa_n + a_r \leq qa_n + b$$

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so that

$$\begin{aligned}\frac{a_m}{m} &\leq \frac{qa_n}{m} + \frac{b}{m}\\ &< \frac{qn(L+\epsilon)}{m} + \frac{b}{m} \xrightarrow{m\to\infty} L+\epsilon\end{aligned}$$

since $qn/m \to 1$ as $m \to \infty$. $\square$

1.1. **Subadditivity for functions on the positive reals.** Here we establish some conditions which guaratee that a function $f\colon (0,\infty) \to \mathbb{R}$ is subadditive:

Proposition 1.3. *(i) If $f(t)/t$ is decreasing on $(0,\infty)$, then $f(t)$ is subadditive.*
(ii) If $f\colon (0,\infty)\to\mathbb{R}$ is concave with $\limsup_{t\to 0} f(t) \geq 0$, then f is subadditive.

Proof. To see (i), we have

$$f(t_1+t_2) = t_1\frac{f(t_1+t_2)}{t_1+t_2} + t_2\frac{f(t_1+t_2)}{t_1+t_2} \leq t_1\frac{f(t_1)}{t_1} + t_2\frac{f(t_2)}{t_2} = f(t_1)+f(t_2)$$

as claimed.

To see (ii), if $f(t)$ is concave, for $0<a<b$, let $0<t<a$ be arbitrary and let α be such that $\alpha t + (1-\alpha)b = a$. Then by concavity, we have

$$f(a) \geq \alpha f(t) + (1-\alpha)f(b) = \alpha f(t) + \frac{a-\alpha t}{b} f(b).$$

Thus

$$f(a) \geq \alpha \limsup_{t\to 0} f(t) + f(b)\limsup_{t\to 0}\frac{a-\alpha t}{b} \geq \frac{a}{b} f(b)$$

so that $f(t)/t$ is decreasing. Then apply (i). $\square$

We can also establish the equivalent statement of Lemma 1.2 for Borel measurable functions $f\colon (0,\infty)\to\mathbb{R}$. The key technical detail is to establish a continuous equivalence of the maximum $\max\{a_i : 1 \leq i \leq n\}$ in the proof of Lemma 1.2.

The proofs of the following lemma and theorem are due to Hille [3]:

Lemma 1.4. *Let $f\colon (0,\infty)\to\mathbb{R}$ be Borel measurable and subadditive. Then f is bounded on any compact subset of $(0,\infty)$.*

Proof. Let $a \in (0,\infty)$ be arbitrary. If $t_1, t_2 \in (0,\infty)$ satisfy $t_1+t_2 = a$, then $f(a) \leq f(t_1)+f(t_2)$. It follows that, with

$$E_a \coloneqq \{t : f(t) \geq f(a)/2, 0 < t < a\},$$

we have $(0,a) = E_a \cup (a - E_a)$ and therefore $m(E_a) \geq a/2$. Suppose for contradiction f is unbounded on some interval (α,β) with $0<\alpha<\beta<\infty$.

If f is not bounded above on (α,β), then there exists a sequence $(t_n)_{n=1}^\infty$ where each $f(t_n) \geq 2n$ and $(t_n)_{n=1}^\infty \to t_0 \in [\alpha,\beta]$. But now each $E_{t_n} = \{t : f(t) \geq n, 0 < t < t_n\} \subset [0,\beta]$ has $m(E_{t_n}) \geq t_n/2 \geq \alpha/2$, a contradiction. Thus f is bounded above on any interval (α,β).

If f is not bounded below on (α, β), then there exists a sequence $(t_n)_{n=1}^{\infty}$ where each $f(t_n) \leq -n$ and $(t_n)_{n=1}^{\infty} \to t_0 \in [\alpha, \beta]$. Let $M = \sup\{f(t) : 2 < t < 5\} < \infty$. Now if $t' \in (2, 5)$, we have $f(t' + t_n) \leq f(t') + f(t_n) \leq M - n$. For sufficiently large n, $(t_0 + 3, t_0 + 4) \subset (t_n + 2, t_n + 5)$ so for each $t \in (t_0 + 3, t_0 + 4)$, we have $f(t) \leq M - n$, a contradiction. Thus f is bounded below on any interval (α, β), and hence bounded below on any compact subset of $(0, \infty)$. $\square$

The previous lemma is the key technical result for the following theorem; the remaining details of the proof are similar to Lemma 1.2.

Theorem 1.5. *Let $f \colon (0, \infty) \to \mathbb{R}$ be Borel measurable and subadditive. Then*

$$\lim_{t \to \infty} \frac{f(t)}{t} = \inf_{t > 0} \frac{f(t)}{t} < \infty.$$

Proof. We first assume $L := \inf_{t>0} \frac{f(t)}{t} > -\infty$; the case $L = -\infty$ follows analgously. For any $\epsilon > 0$, let $b > 0$ be such that $f(b)/b < L + \epsilon$. Now for any $t \geq 2b$, let $n \in \mathbb{N}$ and $b \leq r < 2b$ such that $t = nb + r$. Then

$$\begin{aligned} L \leq \frac{f(t)}{t} &= \frac{f(nb + r)}{t} \leq \frac{nf(b) + f(r)}{t} \\ &\leq \frac{n}{t} \cdot \frac{f(b)}{b} + \frac{f(r)}{t}. \end{aligned}$$

But $r \in [b, 2b]$ and since $\sup\{f(t) : t \in [b, 2b]\} < \infty$ by Lemma 1.4, we have $\lim_{t \to \infty} \frac{f(t)}{t} \leq L + \epsilon$. But $\epsilon > 0$ was arbitrary, so the desired result holds. $\square$

Remark 1.6. *Of course, subadditivity is preserved under isomorphism. Let A and B be abelian semigroups and $f \colon A \to \mathbb{R}$ a subadditive function. If $T \colon A \to B$ is an isomorphism of semigroups, then $g = T \circ f \circ T^{-1} \colon B \to \mathbb{R}$ is also subadditive. For example, submultiplicativity is equivalent to subadditivity by using the map $T(x) = -\log(x)$ as a function from $(0, 1)$ (with multiplication) to $(0, \infty)$ (with addition).*

1.2. **Subadditive functions with multiple arguments.** Next, we establish the following subadditivity result for a certain family of multivariate functions. Note that we must *a priori* assume that our sequence is bounded in the second coordinate.

Lemma 1.7. *Suppose $\{\psi(k, m) : k \in \mathbb{N}, m \in \mathbb{N}\}$ is a family of real numbers such that:*

(i) For each $m \in \mathbb{N}$, there is a $C_m \in \mathbb{R}$ so that $\psi(k, m) \leq C_m$ for all $k \in \mathbb{N}$.

(ii) For all $k, m, n \in \mathbb{N}$,

$$\psi(k, m + n) \leq \frac{m\psi(k, m) + n\psi(k + m, n)}{m + n}.$$

Then

$$\limsup_{m \to \infty} \limsup_{k \to \infty} \psi(k, m) = \lim_{m \to \infty} \limsup_{k \to \infty} \psi(k, m) = \lim_{m \to \infty} \sup_{k \in \mathbb{N}} \psi(k, m)$$

Proof. Note that applying (ii) inductively, we obtain for any $\{m_i\}_{i=1}^{\ell} \subset \mathbb{N}$ and $k \in \mathbb{N}$

$$\psi\big(k, \textstyle\sum_{i=1}^{\ell} m_i\big) \leq \frac{\sum_{i=1}^{\ell} m_i \psi\big(k + \sum_{j=1}^{i-1} m_j, m_i\big)}{\sum_{i=1}^{\ell} m_i}. \tag{1.1}$$

We take the empty sum to be 0.

We establish the first equality. Write $a_m = m \cdot \limsup_{k\to\infty} \psi(k,m)$. Applying (1.1),

$$\begin{aligned} a_{m+n} &= (m+n) \limsup_{k\to\infty} \psi(k, m+n) \\ &\leq (m+n) \limsup_{k\to\infty} \frac{m\psi(k,m) + n\psi(k+m,n)}{m+n} \\ &\leq a_m + a_n. \end{aligned}$$

Thus the sequence $(a_m)_{m=1}^{\infty}$ is subadditive, so the limit of a_m/m exists and is equal to $\inf_{m\in\mathbb{N}} a_m/m$ by Lemma 1.2. Note that the same argument applies with a supremum in place of the limit supremum.

It remains to verify the second equality. Write

$$\beta = \lim_{m\to\infty} \limsup_{k\to\infty} \psi(k,m).$$

The final equality holds trivially if $\beta = -\infty$, so we may assume otherwise. It suffices to show for each $\epsilon > 0$ and all m sufficiently large depending on ϵ and all $k \in \mathbb{N}$,

$$\psi(k,m) \leq \beta + 3\epsilon. \tag{1.2}$$

By the definition of β, there is some m_0 and K so that for all $k \geq K$, $\psi(k, m_0) \leq \beta + \epsilon$. Let

$$C = \max\{C_j : j \in \mathbb{N}, 1 \leq j \leq \max\{m_0, K\}\}.$$

Now let $n \in \mathbb{N}$ be arbitrary and write $n = \ell m_0 + j$ for some $\ell \in \mathbb{N} \cup \{0\}$ and $0 \leq j < m_0$. Applying (1.1), there is some $N \in \mathbb{N}$ so that for all $n \geq N$,

$$\begin{aligned} \psi(k,n) &\leq \frac{m\sum_{i=0}^{\ell-1} \psi(k + im_0, m_0) + j\psi(k + \ell m_0, j)}{n} \\ &\leq \frac{\ell m_0}{n}(\beta + \epsilon) + \frac{j}{n} C \leq \beta + 2\epsilon. \end{aligned}$$

Now let $k \in \{1, \ldots, K\}$ and $m \geq N + K$, and set $j = K - k$. Again applying (1.1),

$$\begin{aligned} \psi(k,m) &\leq \frac{j\psi(k,j) + (m-j)\psi(K, m-j)}{m} \\ &\leq \frac{j}{m} C + \frac{m-j}{m}(\beta + 2\epsilon) \leq \beta + 3\epsilon \end{aligned}$$

for all m sufficiently large since $j \leq K$. This proves (1.2), as required. $\square$

1.3. **Approximate subadditivity and other variants.** Sometimes, it is useful to consider an approximate form of subadditivity.

Definition 1.8. *We say that* $f\colon (0,\infty) \to \mathbb{R}$ *is* approximately subadditive *if there exist constants* $c \in \mathbb{R}$ *and* $r \in (0,\infty)$ *such that*

$$f(x+y+r) \leq f(x) + f(y) + c$$

For example, the following result holds, and the proof is essentially same as Theorem 1.5:

Theorem 1.9. *Let* $f\colon (0,\infty) \to \mathbb{R}$ *be approximately subadditive. Then* $\lim_{t\to\infty} f(t)/t$ *exists and is equal to* $\inf_{t>0} f(t)/t$*.*

We can also consider types of subadditivity for functions of two variables. This result is motivated by the technique used in [2, Prop. 3.1]:

Theorem 1.10. *Let* $f\colon (0,\infty)\times(0,1) \to \mathbb{R}$ *and suppose for any* $\epsilon > 0$*:*

(i) There is $\delta > 0$ *such that whenever* $s,t \in (0,\infty)$ *have* $s/t < \delta$*,*

$$f(t+s, 2\epsilon) \geq f(t,\epsilon),$$

and

(ii) There are $r \in (0,\infty)$ *and* $D > 0$ *such that for any* $\epsilon \in (0,1/2)$ *and* $p \in \mathbb{N}$*, there exists* $N(\epsilon) > 0$ *so that*

$$f(p(t+r), 2\epsilon) \geq D^p (f(t,\epsilon))^p$$

for any $t \geq N(\epsilon)$*.*

Then

$$\lim_{\epsilon\to 0} \limsup_{t\to\infty} \frac{\log f(t,\epsilon)}{t} = \lim_{\epsilon\to 0} \liminf_{t\to\infty} \frac{\log f(t,\epsilon)}{t}.$$

Proof. Let $\epsilon > 0$ be arbitrary and sufficiently small, and set

$$L \coloneqq \limsup_{t\to\infty} \frac{\log f(t,\epsilon)}{t} \qquad\qquad M \coloneqq \liminf_{t\to\infty} \frac{\log f(t,4\epsilon)}{t}.$$

It suffices to show that $L \leq M$. Let $t_0 \geq N(\epsilon)$ be arbitrary and let $t_0 + r \leq s_1 \leq s_2 \leq \cdots$ be a sequence tending to infinity such that

$$\lim_{n\to\infty} \frac{\log f(s_n, 4\epsilon)}{s_n} = M.$$

Now for $n \in \mathbb{N}$ sufficiently large, there exists $p_n \in \mathbb{N}$ and $0 < s \leq t_0 + r$ such that $s_n = p_n(t_0+r) + s$ and $s/(p_n(t_0+r)) < \delta$. Applying (i) and then (ii), we have

$$f(s_n, 4\epsilon) = f(p(t_0+r)+s, 4\epsilon) \geq f(p_n(t_0+r), 2\epsilon) \geq D^{p_n} f(t_0,\epsilon)^{p_n}$$

so that

$$\frac{\log f(s_n,4\epsilon)}{s_n} \geq \frac{\log(D) + \log f(t_0,\epsilon)}{s_n/p_n}.$$

Now, observe that $\lim_{n\to\infty} s_n/p_n = t_0 + r$ so that

$$M \geq \frac{\log(D) + \log f(t_0, \epsilon)}{t_0 + r} \tag{1.3}$$

where $t_0 > 0$ is arbitrary.

Moreover, we observe that $\lim_{t\to\infty} f(t,\epsilon) = \infty$ as a consequence of (ii). Let $(t_n)_{n=1}^{\infty}$ be a sequence tending to infinity with $\lim_{n\to\infty} \frac{\log f(t_n,\epsilon)}{t_n} = L$. Then for each $n \in \mathbb{N}$ with $t_n \geq N(\epsilon)$, we have by (1.3)

$$M \geq \lim_{n\to\infty} \frac{\log D + \log f(t_n, \epsilon)}{t_n + r} = L$$

as required. □

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