On the upper Assouad spectrum

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ABSTRACT. The goal of this note is to provide a simplified proof of the following fact: if E is a non-empty metric space with $\dim_{\mathrm{qA}} E < \infty$, then for $0 \leq \theta < 1$,

$$\overline{\dim}_{\mathrm{A}}^{\theta} E = \sup_{0 \leq \lambda \leq \theta} \dim_{\mathrm{A}}^{\lambda} E.$$

In fact, we will see that this is an immediate consequence of an asymptotic Lipschitz property of a two-scale branching function associated with the set E.

This result was previously proven in [FHH+19] under the additional assumption that E is a subset of Euclidean space, with a more difficult proof.

1. UPPER ASSOUAD SPECTRUM

Let E be a non-empty metric space. Throughout, for $\delta > 0$, $B(x, \delta)$ denotes the open ball with radius δ and $N_\delta(F)$ denotes the least number of open balls of radius δ required to cover a set $F \subset E$.

A two-scale branching function. Before we state the definition of the (upper) Assouad spectrum, we first introduce the *two-scale branching function* associated with E. It is the function β defined for $0 \leq u \leq v$ by

$$\beta(u, v) = \log \sup_{x \in E} N_{2^{-u}}(B(x, 2^{-v})).$$

Here, the base of the logarithm is 2.

We begin with some basic properties of the function β.

Lemma 1.1. *Let E have two-scale branching function β. Then:*

(i) $\beta(u, u) = 0$ *for all* $u \geq 0$.

(ii) $\beta(u, v)$ *is increasing in* u *and decreasing in* v.

(iii) *For all* $0 \leq v \leq w \leq u$,

$$\beta(u, v) \leq \beta(u, w) + \beta(w, v). \tag{1.1}$$

Proof. The fact that $\beta(u, u) = 0$ for $u \geq 0$ is immediate, along with the monotonicity properties. To verify (1.1), let $0 \leq v \leq w \leq u$ be arbitrary. Then to obtain a cover for $B(x, 2^{-v})$, we first cover with balls of radius 2^{-w}, and then cover each ball in the resulting cover with balls of radius 2^{-u}. Therefore

$$N_{2^{-u}}(B(x, 2^{-v})) \leq N_{2^{-w}}(B(x, 2^{-v})) \cdot \sup_{y \in E} N_{2^{-u}}(B(y, 2^{-w})).$$

Taking suprema in x and a logarithm, the claim follows. □

Assouad spectra and quasi-Assouad dimension. Using the two-scale branching function β, let us recall various definitions associated with the Assouad spectrum. First, the *Assouad spectrum* is defined for $0 \leq \theta < 1$ by

$$\dim_{\mathrm{A}}^{\theta} E = \inf\Big\{ s \geq 0 : \exists A \geq 0 \,\forall 0 \leq u \text{ s.t. } \beta(u, \theta u) \leq A + s(1-\theta)u \Big\}.$$

Similarly, the *upper Assouad spectrum* is defined for $0 \leq \theta < 1$ by

$$\overline{\dim}_{\mathrm{A}}^{\theta} E = \inf\Big\{ s \geq 0 : \exists A \geq 0 \,\forall 0 \leq v \leq \theta u \text{ s.t. } \beta(u, v) \leq A + s(u - v) \Big\}.$$

Clearly $\overline{\dim}_{\mathrm{A}}^{\theta} E \geq \dim_{\mathrm{A}}^{\theta} E$, and moreover $\overline{\dim}_{\mathrm{A}}^{\theta} E$ is monotonically increasing in θ. The *quasi-Assouad dimension* is the limit at 1:

$$\dim_{\mathrm{qA}} E = \lim_{\theta \nearrow 1} \overline{\dim}_{\mathrm{A}}^{\theta} E.$$

It is standard to rephrase the above definitions in terms of limits of the function β.

Lemma 1.2. *Let E have two-scale branching function β. Then*

$$\dim_{\mathrm{A}}^{\theta} E = \limsup_{u \to \infty} \frac{\beta(u, \theta u)}{u(1-\theta)}$$

and

$$\overline{\dim}_{\mathrm{A}}^{\theta} E = \limsup_{u \to \infty} \sup_{0 \leq \lambda \leq \theta} \frac{\beta(u, \lambda u)}{u(1-\lambda)}.$$

We can now see that the result claimed in the abstract is simply a justification of the change of the order of the limit in the definition of $\overline{\dim}_{\mathrm{A}}^{\theta} E$. In order to justify this operation, we will prove that the functions $\beta(u, v)$ are uniformly Lipschitz in the variable v up to a sub-linear error term in u.

An asymptotic Lipschitz property. Now let us assume in addition that the quasi-Assouad dimension of E is finite. We prove that the function β satisfies an approximate Lipschitz property.

Lemma 1.3. *Let E have two-scale branching function β, and suppose $\alpha = \dim_{\mathrm{qA}} E < \infty$. Then for all $0 \leq v \leq u$,*

$$\beta(u, v) \leq \alpha(u - v) + o(u).$$

Proof. It suffices to show for all $\varepsilon > 0$ there is a constant $C_\varepsilon \geq 1$ such that for all $0 < r \leq R < 1$,

$$\sup_{x\in E} N_r(B(x,R)) \leq C_\varepsilon r^{-\varepsilon}\left(\frac{R}{r}\right)^{\alpha}. \tag{1.2}$$

Let $\theta < 1$ be sufficiently large so that $(1-\theta)(\alpha+\varepsilon) \leq \varepsilon$. We consider two cases depending on the value of R. If $R \geq r^\theta$, then by definition of the quasi-Assouad dimension there is a constant C_ε (depending on E, θ, and ε) so that

$$N_r(B(x,R)) \leq C_\varepsilon \left(\frac{R}{r}\right)^{\alpha+\varepsilon} \leq C_\varepsilon r^{-\varepsilon}\left(\frac{R}{r}\right)^{\alpha}.$$

In the second inequality we just use that $R < 1$. Otherwise, if $R \leq r^\theta$, since $B(x,R) \subset B(x,r^\theta)$,

$$N_r(B(x,R)) \leq N_r(B(x,r^\theta)) \leq C_\varepsilon \left(\frac{r^\theta}{r}\right)^{\alpha+\varepsilon} \leq C_\varepsilon r^{-(1-\theta)(\alpha+\varepsilon)} \leq C_\varepsilon r^{-\varepsilon}\left(\frac{R}{r}\right)^{\alpha}$$

where the last line follows since $(R/r)^\alpha \geq 1$. Since $x \in E$ was arbitrary, the claim in (1.2) follows.

Taking logarithms and substituting the definition of β, the desired claim follows. □

Using the subadditivity property (1.1), we can convert Lemma 1.3 into an asymptotic Lipschitz property in the second argument of β.

Corollary 1.4. *Let E have two-scale branching function β, and suppose $\alpha = \dim_{\mathrm{qA}} E < \infty$. Then for all $0 \leq w \leq v \leq u$,*

$$0 \leq \beta(u,w) - \beta(u,v) \leq \alpha(v-w) + o(u).$$

Proof. By monotonicity, $\beta(u,w) \geq \beta(u,v)$. Then by (1.1) followed by Lemma 1.3,

$$\beta(u,w) - \beta(u,v) \leq \beta(w,v) \leq \alpha(v-w) + o(u)$$

as claimed. □

Upper Assouad spectra from Assouad spectra. Now, we prove that we can recover the upper Assouad spectrum from the Assouad spectrum.

Theorem 1.5. *Let E be a non-empty metric space with $\dim_{\mathrm{qA}} E < \infty$. Then for $0 \leq \theta < 1$,*

$$\overline{\dim}_{\mathrm{A}}^{\theta} E = \sup_{0 \leq \lambda \leq \theta} \dim_{\mathrm{A}}^{\lambda} E.$$

Proof. Recalling Lemma 1.2, get an increasing sequence $0 < u_n \to \infty$ and $\lambda_n \in [0,\theta]$ such that $\lambda_n \to \lambda \in [0,\theta]$ and

$$\overline{\dim}_{\mathrm{A}}^{\theta} E = \lim_{n\to\infty} \frac{\beta(u_n, \lambda_n u_n)}{u_n(1-\lambda_n)}.$$

Then by Corollary 1.4, for $n \in \mathbb{N}$,

$$|\beta(u_n, \lambda_n u_n) - \beta(u_n, \lambda)| \leq \alpha u_n |\lambda_n - \lambda| + o(u_n). \tag{1.3}$$

Since $\theta < 1$, $(1-\lambda_n)^{-1} \to (1-\lambda)^{-1}$, so

$$\lim_{n\to\infty} \left| \frac{\beta(u_n, \lambda_n u_n)}{u_n(1-\lambda_n)} - \frac{\beta(u_n, \lambda)}{u_n(1-\lambda)} \right| = 0.$$

Therefore $\dim_{\mathrm{A}}^{\lambda} E \geq \overline{\dim}_{\mathrm{A}}^{\theta} E$ as required. □

ACKNOWLEDGEMENTS

The author thanks Vilma Orgoványi for interesting discussions around the contents of this note.

REFERENCES

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