Attainable forms of lower spectra

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ABSTRACT. Let $d \in \mathbb{N}$ and $\varphi\colon (0,1) \to [0,d]$. We prove there exists a set $F \subset \mathbb{R}^d$ whose lower spectrum $\dim_{\mathrm{L}}^\theta F$ satisfies $(1-\theta)\dim_{\mathrm{L}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1)$ if and only if for all $\lambda, \theta \in (0,1)$,

$$\varphi(\theta) \le \varphi(\lambda\theta) - \theta\varphi(\lambda) \le (1-\theta)d.$$

We also obtain a similar classification result for $\underline{\dim}_{\mathrm{L}}^\theta F$.

In contrast to the case for Assouad spectra, it is insufficient to consider homogeneous (or uniform) sets. Instead, we follow the approach introduced by Orgoványi–Rutar in [OR25+] and proceed via a more general classification result for appropriate two-scale branching functions.

Key words and phrases. lower spectrum, dimension classification, branching functions

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CONTENTS

1.   INTRODUCTION

Let X be a metric space. The goal of this paper is to study the general properties of the *lower spectrum*. For $0 < \theta < 1$, it is defined by

$$\dim_{\mathrm{L}}^{\theta} X = \sup\Big\{ s \geq 0 : \exists C > 0 \,\forall 0 < r < 1$$
$$\inf_{x \in X} N_r(B(x, r^{\theta})) \geq C \left(\frac{r^{\theta}}{r}\right)^{s} \Big\}.$$

Here and elsewhere, $B(x, r)$ denotes the closed ball centred at x with radius r and $N_r(E)$ denotes the least number of closed balls of radius r required to cover a set $E \subset X$. The lower spectrum was motivated by the lower dimension $\dim_{\mathrm{L}} X$, which was introduced in [Lar67]. For $0 < \theta < 1$, the inequalities

$$\dim_{\mathrm{L}} X \leq \dim_{\mathrm{L}}^{\theta} X \leq \underline{\dim}_{\mathrm{B}} X$$

always hold, where $\underline{\dim}_{\mathrm{B}} X$ is the lower box (or lower Minkowski) dimension.

The lower spectrum is the dual notion to the (more commonly studied) notion of the *Assouad spectrum*. We will not discuss this in any more detail outside the introduction, but for the convenience of the reader we mention the definition here: for $0 < \theta < 1$, it is defined by

$$\dim_{\mathrm{A}}^{\theta} X = \inf\Big\{ s \geq 0 : \exists C > 0 \,\forall 0 < r < 1$$
$$\sup_{x \in X} N_r(B(x, r^{\theta})) \leq C \left(\frac{r^{\theta}}{r}\right)^{s} \Big\}.$$

Both the Assouad spectrum and the lower spectrum were introduced in [FY18b] and calculated for various families of fractals in [FY18a]; more context can be found in the book [Fra20]. The Assouad spectrum is useful for certain embedding problems [CT23; Fra21], and plays an important role in many "fractal" problems in harmonic analysis [BRRS25; FRS26+; RS23; RSS23]. To the best knowledge of the authors, the lower spectrum has not yet appeared in such contexts. Regardless, the motivation to study the lower spectrum is similar to the motivation for the Assouad spectrum, since they both capture a similar type of scaling information.

In [Rut24], the author obtained a precise answer to the following question: *What functions can appear as the Assouad spectrum of a subset of $\mathbb{R}^d$?* A consequence of such a classification result is that it clarifies the properties of the Assouad spectrum, and it is useful for constructing examples.

The primary goal of this paper is to answer the analogous question for the lower spectrum. A secondary goal is to collect the various properties which have appeared throughout the literature and to clarify the properties of the lower spectrum for general sets.

1.1. Two-scale branching functions and classification.

We now state our main classification result for the lower spectrum.

Theorem A. *Let $d \in \mathbb{N}$ and $\varphi \colon (0,1] \to [0,d]$. Then there exists a set $F \subset \mathbb{R}^d$ such that $(1-\theta) \dim_{\mathrm{L}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1]$ if and only if for all $\lambda, \theta \in (0,1]$,*

$$(1.1) \qquad \varphi(\theta) \leq \varphi(\lambda\theta) - \theta\varphi(\lambda) \leq (1-\theta)d.$$

In the statement, even though we have not formally defined $\dim_{\mathrm{L}}^\theta F$ for $\theta = 1$, we can define $(1-\theta) \dim_{\mathrm{L}}^\theta F$ at 1 by taking a limit (which always yields the value 0).

It was already known that the first inequality of (1.1) holds for every uniformly perfect $F \subset \mathbb{R}^d$ (this follows from [CWC20, Proposition 2.1]) and that the second inequality holds for every $F \subset \mathbb{R}^d$ (this follows from the second part of [Fra20, Proposition 3.1]). We will give a full proof of the necessity of (1.1) (without the uniformly perfect assumption) for completeness. The main novelty in Theorem A is the sufficiency of (1.1).

Since the lower spectrum is defined in a way which seems essentially identical to the Assouad spectrum (except with the various inequalities reversed), one might expect at first that the proof technique from [Rut24] could be modified in a straightforward way to work for the lower spectrum. Perhaps surprisingly, the corresponding classification for the lower spectrum from Theorem A is qualitatively quite different. Indeed, for the Assouad spectrum, while the reverse of the first inequality of (1.1) does appear, the second inequality is replaced with the property that φ is d-Lipschitz, whereas for the lower spectrum the function φ need not be Lipschitz on $(0,1)$.

The reason for this difference is that the lower spectrum has an essential qualitative feature that does not appear for the Assouad spectrum. Consider the space $X = [0, 1/2] \cup \{1\}$. Then, for small r, $\inf_{x \in X} N_r(B(x,1) \cap X) \approx r^{-1}$, whereas $\inf_{x \in X} N_r(B(x, 1/2) \cap X) \approx 1$. In other words, changing the radius of the larger ball by a constant factor can result in an essentially maximal change in the covering number. This type of phenomenon does not occur with the Assouad spectrum (as long as the space X is not too large, for instance, if it is doubling). In particular, attempting to obtain a classification for the lower spectrum using uniform sets (for instance by using homogeneous Moran sets, as used in [BR22; Rut24]) is not possible; see Theorem C below.

Instead, we will approach the classification by studying a more general function, which we call the *infimal two-scale branching function*. Let $\Delta = \{(u, v) \in \mathbb{R}^2 :$

$v \le u\}$. We let $\zeta = \zeta_X \colon \Delta \to [0, \infty]$ denote the function

$$\zeta_X(u, v) = \log \inf_{x \in X} P_{2^{-u}}(B(x, 2^{-v})).$$

Here and elsewhere, the logarithm is in base 2 and for a set E, $P_r(E)$ denotes the maximal cardinality of a subset of E such that distinct points in the subset are separated by distance strictly larger than $4r$. The reason for this choice is that it makes ζ_X superadditive (see Lemma 2.2): for $v \le w \le u$,

(1.2) $$\zeta_X(u, v) \ge \zeta_X(u, w) + \zeta_X(w, v).$$

With other notions of covering (or packing), one only obtains this inequality with an extra error term, and it is convenient not to have to worry about this. The idea to study this branching function originates in [OR25+], which introduces and studies an analogous function with a supremum over covering numbers in place of the infimum over packings.

For subsets $X \subset \mathbb{R}^d$, it is also easy to check for all $v \in \mathbb{R}$ that $u \mapsto \zeta_X(u, v)$ is d-Lipschitz in an approximate sense. In contrast, the function $v \mapsto \zeta_X(u, v)$ need not be anywhere close to Lipschitz. This is already the case for the example $[0, 1/2] \cup \{1\}$ described above.

We obtain a classification result for the possible two-scale branching functions ζ_X. In order to state our result, we require two definitions.

Definition 1.1. Let $\alpha \ge 0$. We let $\mathcal{L}(\alpha)$ denote the functions $f \colon \Delta \to [0, \infty)$ such that the following hold:
  (i) $f(u, u) = 0$ for all $u \ge 0$.
  (ii) For all $v \le w \le u$,

(1.3) $$f(u, v) \ge f(u, w) + f(w, v).$$

  (iii) For all v, the function $u \mapsto f(u, v)$ is α-Lipschitz.

Note that it follows from (1.3) that $f(u, v)$ is an increasing function of u and a decreasing function of v. Also, we note that condition (ii) implies the first inequality in (1.1), and (iii) implies the second inequality in (1.1).

We also require an equivalence relation on the functions $f \colon \Delta \to [0, \infty)$.

Definition 1.2. Let $f, g \colon \Delta \to [0, \infty)$. We write $f \sim g$ if there exists a constant $z \ge 0$ such that for all $0 \le v \le u$,
  • $f(u, v + z) - z \le g(u, v)$ whenever $v \le u - z$, and
  • $g(u, v) \le f(u, v - z) + z$ whenever $v \ge z$.

One can think of the relation $f \sim g$ as stating that f and g are equal up to a fixed additive error term; the actual statement allows some flexibility since we have no control on the function $v \mapsto \zeta_X(u, v)$ at specific points in general (other than knowing that it is monotone). Below, we use the subscript $f \sim_d g$ to indicate that the implicit constants depend only on d.

Theorem B. *Let $d \in \mathbb{N}$.*
  (i) If $X \subset \mathbb{R}^d$, then there is a function $f \in \mathcal{L}(d)$ such that $f \sim_d \zeta_X$.

(ii) If $f \in \mathcal{L}(d)$, then there exists a set $X \subset \mathbb{R}^d$ such that $f \sim_d \zeta_X$.

Let us give a brief outline of the proofs of Theorem A and Theorem B. In §2.1, we establish basic properties of the two-scale branching function ζ_X. Most of the work consists of manipulation to establish the Lipschitz property; this is done in §2.2. By the end of §2.2, we will have proven Theorem B (i). Then in §2.3, we prove Theorem B (ii). (We construct the set as a countable disjoint union of the sets described in §4.2.)

In §3.1, we introduce a normalized limit of branching functions and prove in Proposition 3.2 that the set of limits is precisely those functions which satisfy the inequalities in Theorem A. We then note in §3.2 that the normalized limit is continuous by establishing certain weak Lipschitz properties. Finally, in §3.3, we show using continuity that the limit is unchanged by the equivalence relation in Definition 1.2, and then prove that it corresponds exactly to the lower spectrum of X. The proof of Theorem A then follows by Theorem B.

1.2. Additional classification results. We conclude the introduction by stating two additional classification results.

The first classification result concerns uniform sets. Heuristically, a uniform set is one for which the function $x \mapsto N_r(B(x, R))$ is approximately constant on X, independently of $0 < r \leq R$. In practice, it is only necessary that the error term is sufficiently small as a function of r; see Definition 4.1 for the precise definition.

In [Rut24], it is proven that every Assouad spectrum function can be obtained as the Assouad spectrum of a uniform set (see also [BRT25, §2.4]). In contrast, for the lower spectrum, we saw in the previous section that the essential difficulty involved non-uniform sets. This motivated us to pursue the classification using a more general classification result for two-scale branching functions.

To give a rigorous justification that it would not suffice to use uniform sets in the proof of Theorem A, we prove the following result in §4.

Theorem C. *Let $d \in \mathbb{N}$ and $\varphi \colon (0, 1] \to [0, d]$. Then there is a uniform set $F \subset \mathbb{R}^d$ with $(1 - \theta) \dim_{\mathrm{L}}^\theta F = \varphi(\theta)$ for all $\theta \in (0, 1]$ if and only if φ is d-Lipschitz and*

$$\varphi(\theta) \leq \varphi(\lambda\theta) - \theta\varphi(\lambda)$$

for all $\lambda, \theta \in (0, 1]$.

Comparing this to Theorem A, the key difference is that the second inequality in (1.1) is replaced with the d-Lipschitz property. This results in a strictly smaller family of functions; see Lemma 4.7.

The proof of Theorem C is very similar to the proof of [Rut24, Theorem A]. Here, we write the proof in the language of branching functions, which makes it quite a bit easier to understand the algebra. The details can be found in §4.3.

Our final classification result concerns a monotone version of the lower spectrum. For $0 < \theta < 1$, the *monotone lower spectrum* is given by

$$\underline{\dim}_{\mathrm{L}}^\theta X = \sup\Big\{ s \geq 0 : \exists C > 0 \, \forall 0 < r \leq r^\theta \leq R < 1$$

$$\inf_{x \in X} N_r(B(x, R) \cap X) \geq C \left(\frac{R}{r}\right)^s \Big\}.$$

This was first defined in [CDW17], where it was called the quasi-lower spectrum. For $X \subset \mathbb{R}^d$, it was proven in [HT21, Theorem A.2] that for $0 < \theta < 1$,

$$(1.4) \qquad \underline{\dim}_{\mathrm{L}}^{\theta} X = \inf_{0 < \lambda \leq \theta} \dim_{\mathrm{L}}^{\lambda} X.$$

See also [CWC20, Theorem 1.1]. In Appendix A, we also give a short and simple proof of (1.4) when X is an arbitrary metric space.

Combining Theorem A and (1.4) with short arguments provided in §3.4, we obtain the following classification result for the monotone lower spectrum.

Corollary D. *Let* $d \in \mathbb{N}$ *and* $\varphi \colon (0,1] \to [0,d]$. *Then there exists* $F \subset \mathbb{R}^d$ *such that* $(1 - \theta)\underline{\dim}_{\mathrm{L}}^{\theta} F = \varphi(\theta)$ *for all* $\theta \in (0,1]$ *if and only if* $\theta \mapsto \varphi(\theta)/(1 - \theta)$ *is decreasing and*

$$\varphi(\lambda\theta) - \theta\varphi(\lambda) \leq (1 - \theta)d$$

for all $\lambda, \theta \in (0,1]$.

We also obtain a complete description of the set of functions $\theta \mapsto (1 - \theta)\underline{\dim}_{\mathrm{L}}^{\theta} F$ for subsets $F \subset \mathbb{R}^d$ as infima of an explicit family of functions; see Theorem 3.14 for more detail. As was the case for uniform sets, the family of functions in Corollary D is strictly smaller since the lower spectrum need not be decreasing in general. Such an example was first constructed in [FY18b, §8]; we provide another example with an explicit formula for the lower spectrum in §3.5.2. In Proposition 4.8 we also mention (without proof) an analogous result for the possible monotone lower spectra of uniform sets.

Finally, in Appendix B we provide a self-contained discussion of the inequalities appearing in Theorem A, Theorem C, and Corollary D, with a particular focus on geometric interpretation of the bounds.

1.3. Further research. To conclude the introduction, we suggest some directions for future research.

First, we pose a question concerning the simultaneous classification of the Assouad and lower spectra. Let $d \in \mathbb{N}$ and $\varphi_{\mathrm{A}} \colon (0,1] \to [0,d]$. We recall from [Rut24, Theorem A] that there exists a set $F \subset \mathbb{R}^d$ such that $(1 - \theta)\dim_{\mathrm{A}}^{\theta} F = \varphi_{\mathrm{A}}(\theta)$ for all $\theta \in (0,1]$ if and only if φ_{A} is d-Lipschitz and for all $\lambda, \theta \in (0,1]$,

$$(1.5) \qquad \varphi_{\mathrm{A}}(\theta) + \theta\varphi_{\mathrm{A}}(\lambda) \geq \varphi_{\mathrm{A}}(\lambda\theta) \geq \varphi_{\mathrm{A}}(\theta).$$

However, it turns out that if we know information about $\dim_{\mathrm{A}}^{\theta} F$ or $\dim_{\mathrm{L}}^{\theta} F$, then we can improve (1.1) and (1.5). It is not too difficult to show, for $\varphi_{\mathrm{L}}(\theta) \coloneqq (1 - \theta)\dim_{\mathrm{L}}^{\theta} F$ and $\varphi_{\mathrm{A}}(\theta) \coloneqq (1 - \theta)\dim_{\mathrm{A}}^{\theta} F$, that the following stronger inequalities hold for all $\lambda, \theta \in (0,1]$:

$$(1.6) \qquad \begin{aligned} \varphi_{\mathrm{L}}(\theta) &\leq \varphi_{\mathrm{L}}(\lambda\theta) - \theta\varphi_{\mathrm{L}}(\lambda) \leq \varphi_{\mathrm{A}}(\theta), \\ \theta\varphi_{\mathrm{L}}(\lambda) &\leq \varphi_{\mathrm{A}}(\lambda\theta) - \varphi_{\mathrm{A}}(\theta) \leq \theta\varphi_{\mathrm{A}}(\lambda). \end{aligned}$$

For instance, the upper bound of the first equation in (1.6) is precisely the upper bound in the second part of [Fra20, Proposition 3.1]. We recover the original inequalities from (1.6) since $\varphi_{\mathrm{L}}(\theta) \geq 0$ and $\varphi_{\mathrm{A}}(\theta) \leq d(1 - \theta)$ for all $\theta \in (0,1]$. One might ask if there are no additional joint inequalities.

Question 1.3. *Let $d \in \mathbb{N}$ and suppose $\varphi_{\mathrm{L}}, \varphi_{\mathrm{A}} \colon (0,1] \to [0,d]$ are such that φ_{A} is d-Lipschitz and $\varphi_{\mathrm{L}}, \varphi_{\mathrm{A}}$ satisfy (1.6). Does there exist a set $F \subset \mathbb{R}^d$ such that $(1 - \theta) \dim_{\mathrm{L}}^{\theta} F = \varphi_{\mathrm{L}}(\theta)$ and $(1 - \theta) \dim_{\mathrm{A}}^{\theta} F = \varphi_{\mathrm{A}}(\theta)$ for all $\theta \in (0,1]$?*

One could also ask about the *modified* versions of $\dim_{\mathrm{L}}^{\theta}$ and $\underline{\dim}_{\mathrm{L}}^{\theta}$, defined by

$$\dim_{\mathrm{ML}}^{\theta} X := \sup\{\dim_{\mathrm{L}}^{\theta} Y : Y \subset X\}; \qquad \underline{\dim}_{\mathrm{ML}}^{X} := \sup\{\underline{\dim}_{\mathrm{L}}^{\theta} Y : Y \subset X\}.$$

The motivation for these definitions is that, unlike $\dim_{\mathrm{L}}^{\theta}$ and $\underline{\dim}_{\mathrm{L}}^{\theta}$, the modified versions are monotone for subsets: if $Y \subset X$, then $\dim_{\mathrm{ML}}^{\theta} Y \leq \dim_{\mathrm{ML}}^{\theta} X$ and $\underline{\dim}_{\mathrm{ML}}^{\theta} Y \leq \underline{\dim}_{\mathrm{ML}}^{\theta} X$. It is easy to see that if X is uniform, then $\dim_{\mathrm{ML}}^{\theta} X = \dim_{\mathrm{L}}^{\theta} X$ and $\underline{\dim}_{\mathrm{ML}}^{\theta} X = \underline{\dim}_{\mathrm{L}}^{\theta} X$. In particular, Theorem C shows that many functions can appear as the modified lower spectrum of subsets of $\mathbb{R}^d$. However, the proof of Theorem A does not work for the modified lower spectrum since the sets E_k in the construction may have larger lower spectrum than their union E. This raises the following question.

Question 1.4. *Let $d \in \mathbb{N}$. What is the set of functions $\theta \mapsto \dim_{\mathrm{ML}}^{\theta} F$ for $F \subset \mathbb{R}^d$? What about $\theta \mapsto \underline{\dim}_{\mathrm{ML}}^{\theta} F$?*

2. TWO-SCALE BRANCHING FUNCTIONS

Throughout this section, X is a non-empty metric space. We formally introduce the infimal two-scale branching function and prove Theorem B.

2.1. An infimal two-scale branching function.
We begin with well-separated packings, which we find to be more convenient to work with than covers.

Definition 2.1. For $E \subset X$ and $r > 0$, we let $P_r(E)$ denote the supremum over $k \in \mathbb{N}$ for which there exist points $\{x_1, \ldots, x_k\} \subset E$ such that $B(x_i, 2r) \cap B(x_j, 2r) = \varnothing$ for all $i \neq j$. We call such a collection $\{x_1, \ldots, x_k\}$ a *(strong) r-packing* of E.

Clearly if E is non-empty, then $P_r(E) \geq 1$. Up to a change of radius by a constant factor, packings are equivalent to covers. For $E \subset X$, let $N_r(E)$ denote the least number of closed balls of radius r required to cover E, in which case it is easy to check that

$$(2.1) \qquad N_{4r}(E) \leq P_r(E) \leq N_r(E).$$

We will see in a moment why we find it more convenient to use packings with additional separation instead of covers, or even regular packings.

We now introduce an appropriate two-scale branching function, which we call the *(infimal) two-scale branching function*. Let $\Delta = \{(u,v) \in \mathbb{R}^2 : v \leq u\}$. We let $\zeta = \zeta_X \colon \Delta \to [0, \infty]$ denote the function

$$\zeta_X(u,v) = \log \inf_{x \in X} P_{2^{-u}}(B(x, 2^{-v})).$$

Here and elsewhere, $\log$ denotes the logarithm in base 2. The function ζ satisfies some basic properties.

Lemma 2.2. *Let X be a non-empty metric space. Then the following hold:*
  (i) $\zeta_X(u, u) = 0$ *for all $u \in \mathbb{R}$.*
  (ii) *The function ζ_X is* superadditive: *for $v \leq w \leq u$,*

$$\zeta_X(u, v) \geq \zeta_X(u, w) + \zeta_X(w, v).$$

Proof. The first property is immediate. We will see below that superadditivity follows by (and is our main justification for) our choice of $4r$-separated packings.

Indeed, it suffices to consider the case that $\zeta_X(u, w)$ and $\zeta_X(w, v)$ are finite (or else, by monotonicity, $\zeta_X(u, v) = \infty$). Moreover, if $w \leq u < w + 1$, then $P_{2^{-u}}(B(x, 2^{-w})) = 1$ for all $x \in X$ so $\zeta_X(u, w) = 0$ and the inequality follows by monotonicity. It remains to check the case $u \geq w + 1$. Let $\{x_1, \ldots, x_k\}$ be a maximal 2^{-w}-packing of $B(x, 2^{-w})$. For each x_i, let $\{y_{i,1}, \ldots, y_{i,\ell}\}$ be a 2^{-u}-packing of $B(x_j, 2^{-w})$ where $\ell = \inf_{y \in X} P_{2^{-u}}(y, 2^{-w})$. Consider points $y_{i,j}$ and $y_{i',j'}$. If $i = i'$, then $y_{i,j}$ and $y_{i,j'}$ are 2^{-u+2}-separated for $j \neq j'$. Otherwise, if $i \neq i'$, since the points x_j are 2^{-w+2}-separated, the points $y_{i,j}$ and $y_{i',j'}$ are $2^{-w+1} \geq 2^{-u+2}$-separated. Then $\{y_{i,j}\}$ is a 2^{-u}-packing of $B(x, 2^{-j})$ with cardinality $k \cdot \ell$ so

$$P_{2^{-u}}(x, 2^{-v}) \geq P_{2^{-w}}(x, 2^{-v}) \cdot \inf_{y \in X} P_{2^{-u}}(y, 2^{-w}).$$

Taking an infimum over x followed by a logarithm yields superadditivity. □

2.2. Lipschitz properties.

Unlike the two-scale branching function defined with a supremum, the function ζ_X need not be Lipschitz in both parameters. We recall the example from the introduction: consider the case that $X = [0, 1/2] \cup \{1\}$ and let u be large. For $1 < v \leq u$, $B(1, 2^{-v}) = \{1\}$ so that $\zeta_X(u, v) = 0$. On the other hand, if $v \leq 0$, then $\zeta_X(u, v) = u + O(1)$. Therefore, we see that ζ_X can be highly non-Lipschitz as a function of v.

Regardless, ζ_X must be Lipschitz in an approximate sense as a function of u.

Lemma 2.3. *Suppose $X \subset \mathbb{R}^d$. Then for all $(u, v) \in \Delta$ and $z \geq 0$,*

$$0 \leq \zeta_X(u + z, v) - \zeta_X(u, v) \leq dz + O_d(1).$$

Proof. The first inequality is just monotonicity in u. The other inequality follows from (2.1). Consider a fixed ball $B(x, 2^{-v})$. Then

$$\begin{aligned}
P_{2^{-(u+z)}}(B(x, 2^{-v})) &\leq N_{2^{-(u+z)}}(B(x, 2^{-v})) \\
&\lesssim_d 2^{dz} N_{2^{-u}}(B(x, 2^{-v})) \\
&\lesssim_d 2^{dz} N_{4 \cdot 2^{-u}}(B(x, 2^{-v})) \\
&\leq 2^{dz} P_{2^{-u}}(B(x, 2^{-v})).
\end{aligned}$$

Taking an infimum over $x \in X$ and then a logarithm yields the claimed inequality. □

Now, recall the definition of $\mathcal{L}(\alpha)$ from Definition 1.1. We also let $\mathcal{L}$ denote those functions satisfying Definition 1.1 (i) and (ii). Clearly $\zeta_X \in \mathcal{L}$, and in Lemma 2.3 we showed that ζ_X is approximately a member of $\mathcal{L}(\alpha)$.

For the remainder of this section, we show that we can replace it with a function which is genuinely in $\mathcal{L}(\alpha)$, without changing the function too much.

The first useful technical tool is to observe that $\mathcal{L}(\alpha)$ is closed under pointwise infima.

Proposition 2.4. *Let $\alpha \geq 0$ and $\mathcal{F} \subset \mathcal{L}(\alpha)$. Then $\inf_{f \in \mathcal{F}} f \in \mathcal{L}(\alpha)$.*

Proof. Write $g = \inf_{f \in \mathcal{F}} f$. Clearly $g(u, u) = 0$ for all $u \in \mathbb{R}$. Throughout the proof, let $\varepsilon > 0$ be arbitrary.

To check superadditivity, let $v \leq u$ be arbitrary and get $f \in \mathcal{F}$ such that $f(u, v) \leq g(u, v) + \varepsilon$. Then,

$$g(u, w) + g(w, v) \leq f(u, w) + f(w, v) \leq f(u, v) \leq g(u, v) + \varepsilon.$$

To check the Lipschitz property, let $v \leq w \leq u$ be arbitrary and get $f \in \mathcal{F}$ such that $f(w, v) \leq g(w, v) + \varepsilon$ so that Then

$$g(u, v) \leq f(u, v) \leq f(w, v) + \alpha(u - w) \leq g(w, v) + \alpha(u - w) + \varepsilon$$

yielding the Lipschitz property. $\qquad\square$

With the above result in mind, we can construct functions $f \in \mathcal{L}(\alpha)$ as an infimum over explicit functions which are easier to describe. Let $z \in \mathbb{R}$ and suppose $g \colon \mathbb{R} \to \mathbb{R}$ is an increasing α-Lipschitz function with $g(u) = 0$ for all $u \leq z$. We define a function

$$(2.2) \qquad h_{g,z}(u, v) = \begin{cases} g(u) - g(v) & : z \leq v, \\ \alpha(u - v) & : z > v. \end{cases}$$

These are the maximal functions in $\mathcal{L}(\alpha)$ which coincide with g on the horizontal strip $\{(u, z) : u \geq z\} \subset \Delta$.

Lemma 2.5. *Let $z \in \mathbb{R}$ and suppose $g \colon \mathbb{R} \to \mathbb{R}$ is a non-negative increasing α-Lipschitz function with $g(z) = 0$. Then $h_{g,z} \in \mathcal{L}(\alpha)$. Moreover if $f \in \mathcal{L}$ and $\eta \colon \mathbb{R} \to [0, \infty)$ are such that $f(u, z) - \eta(u) \leq g(u) \leq f(u, z)$ for all $u \geq z$, then*

$$h_{g,z}(u, v) \geq f(u, v) - \eta(u)$$

for all $(u, v) \in \Delta$.

Proof. We first check that $h_{g,z} \in \mathcal{L}(\alpha)$.

Let $A_z = \{(u, v) \in \Delta : z \leq v\}$ and $B_z = \{(u, v) \in \Delta : z > v\}$. Clearly $h_{g,z}(u, u) = 0$ for all $u \in \mathbb{R}$, $u \mapsto h_{g,z}(u, v)$ is α-Lipschitz, and $u \mapsto h_{g,z}(u, v)$ is increasing as a function of u. It remains to check superadditivity. Let $v \leq w \leq u$. If $z \leq v$ or $z > w$, then (v, w) and (w, u) both lie in either A_z or B_z, in which case the inequality is an easy computation. In the remaining case, $v < z \leq w \leq u$ so that

$$h_{g,z}(u, v) = \alpha(u - v) \geq \alpha(w - v) + g(w) - g(u) = h_{g,z}(u, w) + h_{g,z}(w, v).$$

Here, we used that g is α-Lipschitz so $g(w) - g(u) \leq \alpha(u - w)$.

Finally, let $f \in \mathcal{L}$ be arbitrary and let η be defined as in the statement. Let $(u, v) \in \Delta$ be arbitrary. If $z \leq v \leq u$, then

$$f(u, v) \leq f(u, z) - f(v, z) \leq g(u) - g(v) + \eta(u) = h_{g,z}(u, v) + \eta(u).$$

Otherwise, $z > v$, in which case

$$f(u, v) \leq \alpha(u - v) = h_{g,z}(u, v)$$

since $f(v, v) = 0$ and f is α-Lipschitz in the first variable. $\qquad\square$

Using this lemma we can replace functions which are approximately α-Lipschitz in the first variable with elements of $\mathcal{L}(\alpha)$.

Proposition 2.6. *Let $f \in \mathcal{L}$ and suppose $\eta \colon \mathbb{R} \to [0, \infty)$ is such that for all $v \leq w \leq u$,*

$$f(u, v) - f(w, v) \leq \alpha(u - w) + \eta(u).$$

Then there exists a function $g \in \mathcal{L}(\alpha)$ such that

$$f(u, v) - \eta(u) \leq g(u, v) \leq f(u, v)$$

for all $(u, v) \in \Delta$.

Proof. First, fix a value $z \in \mathbb{R}$ and consider the function $f_z(u) = f(u, z)$. Observe that $f_z(u) = 0$, and let $\phi_z \leq f_z$ be the maximal increasing α-Lipschitz function with $\phi_z(u) = 0$ for all $u \leq z$. Note for $z \leq w \leq u$ that

$$f_z(w) \geq f_z(u) - \alpha(u - v) - \eta(u).$$

In particular, the non-negative increasing α-Lipschitz function which takes value $\max\{0, f_z(u) - \eta(u)\}$ at u and is otherwise as small as possible is bounded above by f_z. Therefore, for all $u \geq z$,

$$f_z(u) - \eta(u) \leq \phi_z(u) \leq f_z(u).$$

Finally, set $g = \inf_{z \in \mathbb{R}} h_{\phi_z, z} \in \mathcal{L}(\alpha)$ by Proposition 2.4.

Clearly $g \leq f$ by construction. On the other hand, it follows from Lemma 2.5 that $h_{\phi_z, z}(u, v) \geq f(u, v) - \eta(u)$ for all $(u, v) \in \Delta$. Therefore

$$g(u, v) \geq f(u, v) - \eta(u)$$

as required. $\qquad\square$

By combining Lemma 2.3 and Proposition 2.6, we have proven the following.

Restatement (of Theorem B (i)). *Let $X \subset \mathbb{R}^d$ be non-empty. Then there exists an $f \in \mathcal{L}(d)$ such that $f \sim_d \zeta_X$.*
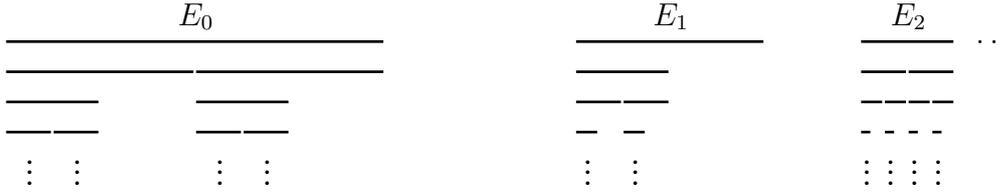
FIGURE 1. A depiction of the set constructed in the proof of Theorem B (ii). Each set E_k is constructed as an intersection of nested families of dyadic cubes, following the procedure in §4.2, with the subdivision sequence of each E_k depending on f_k. The spacing between the consecutive sets is proportional to the diameter of E_k. The sets E_k for $k \geq 3$ are omitted.

2.3. Attainable two-scale branching functions.

Now, we proceed with the proof of the second half of Theorem B. Recall the equivalence relation introduced in Definition 1.2 in the introduction.

We also introduce a definition highlighting the precise properties that we will require for an arrangement of sets in our construction.

Definition 2.7. A *geometric sequence* of compact sets is a sequence of sets $\{E_k\}_{k=0}^{\infty}$ for which there is a constant $m \geq 0$ and points $x_k \in E_k$ so that, for all integers $k \geq 0$,

(i) $E_k \subset B(x_k, 2^{-k})$,
(ii) $E_j \cap B(x_k, 2^{-k}) = \varnothing$ for all $j \neq k$,
(iii) $E_k \subset B(0, 2^{-k+m})$.

Given non-empty sets $\{F_k\}_{k=0}^{\infty}$ with $\operatorname{diam} F_k \leq 2^{-k}$, it is easy to see that there are translations $t_k \in \mathbb{R}^d$ so that $\{F_k + t_k\}_{k=0}^{\infty}$ is a geometric sequence.

Restatement (of Theorem B (ii)). *Let $d \in \mathbb{N}$ and $f \in \mathcal{L}(d)$ be arbitrary. Then there exists a non-empty compact set $X \subset \mathbb{R}^d$ such that $f \sim_d \zeta_X$.*

Proof. First, for each integer $k \geq 0$, let f_k denote the function $f_k(u) = f(u, k)$ for $u \geq k$, and $f_k(u) = 0$ otherwise. Since $f \in \mathcal{L}(d)$, f_k is an increasing d-Lipschitz function. Therefore, by Proposition 4.5 combined with an appropriate translation, there exists a set $E_k \subset B(x_k, 2^{-k})$ with branching function

$$\zeta_{E_k}(u, v) = f_k(u) - f_k(v) + O(1)$$

such that $\{E_k\}_{k=0}^{\infty}$ is a geometric sequence with respect to the points x_k and constant $m \geq 0$ depending only on d. Let

$$X := \{0\} \cup \bigcup_{k=0}^{\infty} E_k.$$

The upper bound is immediate: for each $k \geq 0$,

$$\zeta_X(u, k) \leq \log P_{2^{-u}}(X \cap B(x_k, 2^{-k})) = f_k(u) - f_k(k) + O(1) = f(u, k) + O(1).$$

For the lower bound, let $x \in X$ be arbitrary. First suppose either $x = 0$ or $x \in E_n$ for some $n \geq k$. Since $E_n \subset B(x, 2^{-n+m}) \subset B(x, 2^{-k+m})$, it follows that $E_k \subset B(x, 2^{-k+m+1})$ and therefore

$$
\begin{aligned}
\log P_{2^{-u}}(X \cap B(x, 2^{-k+m+1})) &\geq \log P_{2^{-u}}(E_k \cap B(x, 2^{-k+m+1})) \\
&= f_k(u) - f_k(k - m - 1) + O(1) \\
&= f(u, k) + O(1).
\end{aligned}
$$

Otherwise, $x \in E_n$ for some $0 \leq n \leq k$. Then using superadditivity of f,

$$
\begin{aligned}
\log P_{2^{-u}}(X \cap B(x, 2^{-k+m+1})) &\geq \log P_{2^{-u}}(E_n \cap B(x, 2^{-k+m+1})) \\
&= f_n(u) - f_n(k - m - 1) + O(1) \\
&= f(u, n) - f(k, n) + O(1) \\
&\geq f(u, k) + O(1).
\end{aligned}
$$

Since this handles all possible cases of $x \in X$, we have shown that

$$
\zeta_X(u, k - m - 1) \geq f(u, k) + O(1)
$$

for all integers $k \geq 0$. Combined with the upper bound, it follows that $\zeta_X \sim f$, as required. $\square$

Remark 2.8. In the above proof, we construct a compact set since we are only interested in small scales (that is, the values of $\zeta_X(u, v)$ for $0 \leq v \leq u$). If one is also interested in large scales, one can instead take a union $\bigcup_{k \in \mathbb{Z}} E_k$ where the sets E_k for $k < 0$ are constructed starting at a dyadic cube of side-length 2^{-k} which is much larger than 1. The resulting set will be unbounded, but the proof is the same.

3.   CLASSIFICATION FOR THE LOWER SPECTRUM

In the previous section, we proved the classification for the two-scale branching functions ζ_X. In this section, we relate the function ζ_X to the lower spectrum and prove Theorem A and Corollary D.

3.1. A normalized limit of branching functions.

Let $f \colon \Delta \to [0, \infty]$ be arbitrary. We let $\Lambda(f)$ denote the function defined for $\theta \in (0, 1]$ by the rule

$$
\Lambda(f)(\theta) = \liminf_{u \to \infty} \frac{f(u, \theta u)}{u}.
$$

We will see that the range of Λ restricted to $\mathcal{L}(\alpha)$ is the following space of functions.

Definition 3.1. Let $\alpha \geq 0$. We let $\mathcal{H}(\alpha)$ denote the functions $\varphi \colon (0, 1] \to [0, \alpha]$ such that the following hold:
  (S) *Superadditive:* For all $\lambda, \theta \in (0, 1]$,

$$
\varphi(\lambda\theta) \geq \varphi(\theta) + \theta\varphi(\lambda).
$$

(W) *Weak Lipschitz:* For all $\lambda, \theta \in (0, 1]$,

$$\varphi(\lambda\theta) \leq (1 - \theta)\alpha + \theta\varphi(\lambda).$$

Let us make some simple observations about the functions in $\mathcal{H}(\alpha)$.

1. It follows from (S) that each $\varphi \in \mathcal{H}(\alpha)$ is decreasing, since $\theta\varphi(\lambda) \geq 0$. In particular, we may define $\varphi(0) := \lim_{\theta \to 0} \varphi(\theta)$. Observe that the inequalities (S) and (W) are satisfied for all $\lambda, \theta \in [0, 1]$ with this definition.
2. It follows that $\varphi(1) = 0$ by taking $\theta = 1$ in (S).
3. By (W) with $\lambda = 1$, $\varphi(\theta) \leq \alpha(1 - \theta)$. In particular, (W) also implies that $\varphi(1) = 0$.
4. One can interpret (W) as a weak type of convexity at 0: for all $\lambda \in [0, 1]$, the graph of φ lies below the line segment connecting $(0, \alpha)$ and $(\lambda, \varphi(\lambda))$.
5. Using the above observation, it follows that (W) holds if and only if the function

$$\theta \mapsto \frac{\alpha - \varphi(\theta)}{\theta}$$

is decreasing.

To motivate the space $\mathcal{H}(\alpha)$, we observe that it is precisely the class of normalized limits of elements of $\mathcal{L}(\alpha)$.

Proposition 3.2. *Let $\alpha \geq 0$. Then $\Lambda(\mathcal{L}(\alpha)) = \mathcal{H}(\alpha)$.*

Proof. First let $f \in \mathcal{L}(\alpha)$. We already saw earlier that $\varphi = \Lambda(f)$ is a function from $(0, 1]$ to $[0, \alpha]$. Let us verify the inequalities. First, (S) is a consequence of superadditivity of f:

$$f_u(\lambda\theta) \geq f_u(\lambda) + \theta f_{\theta u}(\lambda)$$

and taking a limit infimum in u yields the desired inequality.

Next, (W) is a consequence of the horizontal Lipschitz property. Fix $\theta > 0$. Let $\varepsilon > 0$ and, by definition of $\Lambda(f)$, get arbitrarily large $u > 0$ so that

$$f(\theta u, \lambda\theta u) \leq \theta u(\varphi(\lambda) + \varepsilon).$$

Then by the α-Lipschitz property,

$$f(u, \lambda\theta u) \leq \alpha(1 - \theta)u + f(\theta u, \lambda\theta u) \leq \alpha(1 - \theta)u + \theta\varphi(\lambda) + u\theta\varepsilon.$$

Since this holds for arbitrarily large u, dividing by u and taking a limit infimum,

$$\varphi(\lambda\theta) \leq (1 - \theta)\alpha + \theta\varphi(\lambda) + \theta\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, (W) follows.

Conversely, let $\varphi \in \mathcal{H}(\alpha)$ and define $f: \Delta \to [0, \infty)$ by the rule $f(u, v) = u\varphi(v/u)$. We first check that $f \in \mathcal{L}(d)$. Let $0 \leq v \leq w \leq u$ be arbitrary, let $\theta = w/u$, and let $\lambda = v/w$. Then using (S),

$$f(u, v) = u\varphi(\lambda\theta) \geq u\varphi(\theta) + u\theta\varphi(\lambda)) = f(u, w) + f(w, v),$$

yielding superadditivity of f. To verify the horizontal Lipschitz property, we again compute by (W) that

$$f(u, v) = u\varphi(\lambda\theta) \leq u\alpha(1 - \theta) + u\theta\varphi(\lambda) = \alpha(u - w) + f(w, v)$$

which, after re-arranging, shows that $u \mapsto f(u, v)$ is α-Lipschitz. Therefore, Λ is surjective. $\qquad\square$

3.2. Continuity and Lipschitz properties.

For the Assouad spectrum, the corresponding space of functions is the set $\mathcal{G}(\alpha)$ of functions $\varphi\colon [0, 1] \to [0, \alpha]$ which are α-Lipschitz, decreasing, and such that for all $\lambda, \theta \in [0, 1]$, the following analogue of (S) holds:

$$\varphi(\lambda\theta) \leq \varphi(\theta) + \theta\varphi(\lambda).$$

When $\alpha = d$ is an integer, this is the family $\{\theta \mapsto (1 - \theta)h(\theta) : h \in \mathcal{A}_d\}$ where $\mathcal{A}_d$ is defined in [Rut24]; and it is the same as the family $\mathcal{G}(\alpha)$ from [OR25+, Definition 1.2].

A key difference in the setting of lower spectra is that the α-Lipschitz property is replaced by property (W). We note that this property is strictly weaker; see §4.3 for more discussion. For now, let us note that the best Lipschitz property which we can obtain is the following.

Lemma 3.3. *Let $\alpha \geq 0$ and $\varphi \in \mathcal{H}(\alpha)$. Then for all $\lambda > 0$, φ is $\lambda^{-1}(\alpha - \varphi(\lambda))$-Lipschitz on the interval $[\lambda, 1]$. In particular, φ is continuous on $(0, 1]$.*

Proof. Fix $\theta \in [\lambda, 1]$. By (W), for all $0 < \theta' \leq \theta$, $\varphi(\theta')$ is bounded above by the line segment connecting $(0, \alpha)$ and $(\theta, \varphi(\theta))$. The negative of the slope of this segment is

$$\frac{\alpha - \varphi(\theta)}{\theta} \leq \frac{\alpha - \varphi(\lambda)}{\lambda}.$$

Since φ is decreasing, the other side of the Lipschitz inequality is trivial, completing the proof. $\qquad\square$

With Lemma 3.3 in mind, we embed $\mathcal{H}(\alpha) \subset C([0, 1])$ where we define $\varphi(0) = \lim_{\theta \to 0} \varphi(\theta)$. As discussed before, the limit exists since φ is decreasing and bounded above by α.

3.3. Classification of lower spectra.

In this section, we complete the proof of Theorem A.

One direction of the proof requires showing that the lower spectrum of X (after normalization by a factor $(1 - \theta)$) is an element of $\mathcal{H}(\alpha)$. In the introduction, we introduced the equivalence relation $\sim$ on functions in $\mathcal{L}$. Below, we introduce a somewhat more general equivalence relation since it more precisely captures the type of errors which we are able to ignore.

Definition 3.4. Let $f, g\colon \Delta \to [0, \infty)$ be arbitrary functions. We say that $f \asymp g$ if there exist a function $\eta\colon [0, \infty) \to [0, \infty)$ with $\lim_{u\to\infty} u^{-1}\eta(u) = 0$ such that for all $0 \le v \le u$:

- $f(u, v + \eta(u)) - \eta(u) \le g(u, v)$ whenever $v \le u - \eta(u)$, and
- $g(u, v) \le f(u, v - \eta(u)) + \eta(u)$ whenever $v \ge \eta(u)$.

Example 3.5. If $f, g \in \mathcal{L}$ and $|f(u, k) - g(u, k)| \le \eta(u)$ for all $0 \le k \le u$ with $k \in \mathbb{Z}$ and $\lim_{u\to\infty} u^{-1}\eta(u) = 0$, then $f \asymp g$ with function $1 + \eta(u)$.

We first observe that the normalized limit Λ is unchanged by the above equivalence relation.

Lemma 3.6. *Suppose $f \in \mathcal{L}(\alpha)$ and $g\colon \Delta \to \mathbb{R}$ are such that $f \asymp g$. Then $\Lambda(f) = \Lambda(g)$.*

Proof. Let $\eta\colon [0, \infty) \to [0, \infty)$ be the functions implicit in the relation $f \asymp g$, and we recall that $\lim_{u\to\infty} u^{-1}\eta(u) = 0$. Also, let $\varphi = \Lambda(f) \in \mathcal{H}(\alpha)$.

Fix $\theta \in (0, 1]$ and write $\varepsilon(u) = u^{-1}\eta(u)$. Since $f \asymp g$, if $\theta < 1$, for all $u > 0$ sufficiently large

$$f(u, (\theta + \varepsilon(u))u) - \eta(u) \le g(u, \theta u) \le f(u, (\theta - \varepsilon(u))u) + \eta(u);$$

and for $\theta = 1$,

$$0 \le g(u, u) \le f(u, (1 - \varepsilon(u))u) + \eta(u).$$

Since φ is continuous by Lemma 3.3, dividing by u and taking a limit infimum gives that $\Lambda(g)(\theta) = \varphi(\theta)$. $\qquad\square$

Now, recall that the usual definition of the lower spectrum from the introduction uses packings instead of covers. We also recall the equivalence of packings and covers from (2.1). We obtain the following alternative formula for $\dim_{\mathrm{L}}^{\theta} F$.

In the below statement, even though $\dim_{\mathrm{L}}^{\theta} F$ is *a priori* not defined for $\theta = 1$, since $\dim_{\mathrm{L}}^{\theta} F \le d$ for $F \subset \mathbb{R}^d$, it makes sense to write $(1 - \theta) \dim_{\mathrm{L}}^{\theta} F$ to mean 0 when $\theta = 1$.

Corollary 3.7. *Suppose $F \subset \mathbb{R}^d$. Then for all $\theta \in (0, 1]$,*

$$(1 - \theta) \dim_{\mathrm{L}}^{\theta} F = \Lambda(\zeta_F).$$

In particular, $\theta \mapsto (1 - \theta) \dim_{\mathrm{L}}^{\theta} F \in \mathcal{H}(d)$.

Proof. Define the function

$$\gamma(u, v) = \log \inf_{x \in F} N_{2^{-u}}(F \cap B(x, 2^{-v})).$$

By (2.1), $\gamma \asymp \zeta_F$, and by Theorem B (i), there is a function $f \in \mathcal{L}(d)$ such that $\zeta_F \asymp f$. Therefore $\gamma \asymp f$ and thus

$$\Lambda(\gamma) = \Lambda(f) = \Lambda(\zeta_F)$$

by Lemma 3.6. Moreover, $\Lambda(f) \in \mathcal{H}(d)$ by Proposition 3.2, and therefore $\Lambda(\zeta_F) \in \mathcal{H}(d)$.

It remains to show that $(1 - \theta) \dim_{\mathrm{L}}^{\theta} F = \Lambda(\gamma)(\theta)$ for $\theta \in (0, 1)$. Fix $\theta \in (0, 1)$. Unpacking definitions, $\dim_{\mathrm{L}}^{\theta} F$ is equivalently the supremum over values $s \geq 0$ for which there exists a constant $m_s \geq 0$ such that for all $u \geq 0$,

$$(3.1) \qquad \gamma(u, \theta u) \geq s(1 - \theta)u - m_s.$$

On one hand, if $s \geq 0$ and $m_s \geq 0$ satisfy (3.1), dividing by u and taking a limit infimum, it is immediate that $\Lambda(\gamma) \geq s$. On the other hand, suppose t is such that for all $m \geq 0$ there is a $u_m \geq 0$ so that

$$\gamma(u_m, \theta u_m) \leq t(1 - \theta)u_m - m.$$

Since γ is non-negative, we must have $\lim_{m \to \infty} u_m = \infty$ and therefore $\Lambda(\gamma) \leq t$. We conclude that $(1 - \theta) \dim_{\mathrm{L}}^{\theta} F = \Lambda(\gamma)(\theta)$. $\qquad\square$

We can now complete the proof of our main result.

Restatement (of Theorem A). *Let $d \in \mathbb{N}$. Then there exists a set $F \subset \mathbb{R}^d$ such that $(1 - \theta) \dim_{\mathrm{L}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0, 1]$ if and only if $\varphi \in \mathcal{H}(d)$.*

Proof. Recall that we saw in Corollary 3.7 that $\theta \mapsto (1 - \theta) \dim_{\mathrm{L}}^{\theta} F \in \mathcal{H}(d)$, so we proceed with the converse. Fix $\varphi \in \mathcal{H}(d)$, and using surjectivity of Λ from Proposition 3.2 followed by Theorem B (ii), we get a set $F \subset \mathbb{R}^d$ such that $\Lambda(f) = \varphi$ and $f \asymp \zeta_F$. By construction, $\Lambda(f) = \varphi$, and by Lemma 3.6, $\Lambda(f) = \Lambda(\zeta_F)$. Therefore $(1 - \theta) \dim_{\mathrm{L}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0, 1]$, as required. $\qquad\square$

Remark 3.8. The set F constructed in Theorem A can in fact be taken to be a countable union of non-trivial closed dyadic cubes. To prove this, one can modify the function f as follows. Let $(n_k)_{k=1}^{\infty}$ be any sequence such that $\lim_{k \to \infty} k/n_k = 0$. For each d-Lipschitz horizontal strip f_k, define a new d-Lipschitz function $g_k \in \mathcal{Y}(d)$ by the rule

$$g_k(u) = \begin{cases} f_k(u) & : 0 \leq u \leq n_k \\ f_k(n_k) + d(u - n_k) & : u \geq n_k \end{cases}$$

Since g_k has constant slope d on $[n_k, \infty)$, the corresponding set provided by §4.2 will be a finite union of dyadic cubes. Now, set $\phi_k = h_{g_k, k}$ where we recall that $h_{g_k, k}$ was defined in (2.2). By definition, $\phi_k(u, k) = f(u, k)$ for $u \leq n_k$. We now follow a similar argument to Lemma 2.5 to show that $\phi_k(u, v) \geq f(u, v)$. Certainly, $\phi_k(u, v) \geq f(u, v)$ for $v \leq k$. Otherwise, consider $k \leq v \leq u$.

- If $n_k \leq v \leq u$, then by the horizontal d-Lipschitz property, $g_k(u) - g_k(v) = d(u - v) \geq f(u, v)$.
- If $v \leq n_k \leq u$, by superadditivity and the horizontal d-Lipschitz property,

$$g_k(u) - g_k(v) = d(u - n_k) + f_k(n_k) - f_k(v) \geq d(u - n_k) + f(n_k, v) \geq f(u, v).$$

- If $v \le u \le n_k$, by superadditivity

$$g_k(u) - g_k(v) = f_k(u) - f_k(v) \ge f(u,v).$$

Therefore, if we define $g = \min_k \phi_k \in \mathcal{L}(d)$ then $g \ge f$, and moreover if $\theta > 0$ is fixed, then $f \sim g$ on the set $\{(u,v) : N \le v \le \theta u\}$ where N is sufficiently large so that $k/n_k < \theta$ for all $k \ge N$. Therefore $\Lambda(g) = \Lambda(f)$, so applying (the proof of) Theorem B (ii), the resulting set is a countable union of dyadic cubes and has the desired lower spectrum.

3.4. Monotone lower spectra. Recall that the monotone lower spectrum was defined in §1.2, and that it is related to the usual spectrum by

(3.2) $$\underline{\dim}_{\mathrm{L}}^{\theta} X = \inf_{0 < \lambda \le \theta} \dim_{\mathrm{L}}^{\lambda} X$$

for $0 < \theta < 1$. This relationship allows us to provide a classification of the functions $\theta \mapsto (1-\theta)\underline{\dim}_{\mathrm{L}}^{\theta} X$, where $X \subset \mathbb{R}^d$. The appropriate space of functions is the following.

Definition 3.9. Let $\alpha \ge 0$. We let $\mathcal{M}(\alpha)$ denote the functions $\varphi \colon (0,1] \to [0,\alpha]$ such that (W) holds and the following condition holds:
 (M) *Monotone:* The function $\theta \mapsto \varphi(\theta)/(1-\theta)$ is decreasing.

Remark 3.10. Property (M) has a useful geometric interpretation: it says that for any $0 < \theta \le 1$, φ is bounded above on $[\theta, 1]$ by the line segment joining $(\theta, \varphi(\theta))$ and $(1,0)$.

We also recall the geometric interpretation of (W): for any $0 < \theta \le 1$, φ is bounded above on $[0, \theta]$ by the line segment joining $(0, \alpha)$ and $(\theta, \varphi(\theta))$.

The main difference from the space $\mathcal{H}(\alpha)$ is that the superadditivity assumption (S) is replaced by the monotonicity assumption (M). We observe that the monotonicity assumption is stronger.

Lemma 3.11. *Let $\alpha \ge 0$. Then $\mathcal{M}(\alpha) \subset \mathcal{H}(\alpha)$ and the inclusion is strict for $\alpha > 0$.*

Proof. Let $\lambda, \theta \in (0,1]$. By (M),

$$\varphi(\theta) \le \frac{\varphi(\lambda\theta)}{1-\lambda\theta}(1-\theta)$$

which, after rearranging and again applying (M), yields

$$\frac{\varphi(\lambda\theta) - \varphi(\theta)}{\theta(1-\lambda)} \ge \frac{\varphi(\lambda\theta)}{1-\lambda\theta} \ge \frac{\varphi(\lambda)}{1-\lambda}.$$

Rearranging, we obtain the required inequality (S).

To see that the inclusion is strict for $\alpha > 0$, see the example constructed in §3.5.2 below. $\qquad\square$

Remark 3.12. There is a natural geometric interpretation of the above computation. Rearranging (S),

$$\frac{\varphi(\lambda\theta) - \varphi(\theta)}{\theta - \lambda\theta} \geq \frac{\varphi(\lambda)}{1 - \lambda}.$$

The left hand side is the negative of the slope of the line segment from $(\lambda\theta, \varphi(\lambda\theta))$ to $(\theta, \varphi(\theta))$, and the right hand side is the negative of the slope of the line segment from $(\lambda, \varphi(\lambda))$ to $(1, 0)$. Recalling Remark 3.10, we see that the monotonicity assumption (M) is stronger.

As the final component of our classification, we introduce a special class of functions using which we can use to give an alternative definition of $\mathcal{M}(\alpha)$. Given $\alpha \geq 0$, $\lambda \in (0, 1]$, and $0 \leq t \leq \alpha(1 - \lambda)$, we define the function

$$\phi_{\alpha,\lambda,t}(\theta) = \begin{cases} \alpha + \frac{\theta}{\lambda}(t - \alpha) & : 0 < \theta \leq \lambda, \\ t\frac{1-\theta}{1-\lambda} & : \lambda \leq \theta \leq 1. \end{cases}$$

This is the function with two affine parts connecting the points $(0, \alpha)$, (λ, t), and $(1, 0)$. The constraint on t ensures that the negative of the slope of $\phi_{\alpha,\lambda,t}$ on $[0, \lambda]$ is greater than or equal to the negative of the slope on $[\lambda, 1]$. In particular, with Remark 3.10 in mind, the following lemma is immediate.

Lemma 3.13. *Let $\alpha \geq 0$. Then $\phi_{\alpha,\lambda,t} \in \mathcal{M}(\alpha)$ for all $\lambda \in (0, 1]$ and $0 \leq t \leq \alpha(1 - \lambda)$. Moreover, $\varphi \in \mathcal{M}(\alpha)$ if and only if for all $\lambda \in (0, 1]$, $\varphi \leq \phi_{\alpha,\lambda,\varphi(\lambda)}$.*

We can now state and prove our main result for the monotone lower spectrum. In particular, this implies Corollary D.

Theorem 3.14. *Let $d \in \mathbb{N}$ and $\varphi \colon [0, 1] \to [0, d]$. Then the following are equivalent.*
  (i) *$\varphi \in \mathcal{M}(d)$.*
 (ii) *There exists $F \subset \mathbb{R}^d$ such that $(1 - \theta)\underline{\dim}_{\mathrm{L}}^\theta F = \varphi(\theta)$ for all $\theta \in (0, 1]$.*
(iii) *φ is an infimum of a family of functions of the form $\phi_{d,\lambda,t}$ for $\lambda \in (0, 1]$ and $0 \leq t \leq \alpha(1 - \lambda)$.*

  Proof. To see that (i) implies (ii), since $\mathcal{M}(d) \subset \mathcal{H}(d)$, we may apply Theorem A to get a set $F \subset \mathbb{R}^d$ such that $(1 - \theta)\dim_{\mathrm{L}}^\theta F = \varphi(\theta)$. Then by (3.2), since $\theta \mapsto \varphi(\theta)/(1 - \theta)$ is decreasing, it follows that $\underline{\dim}_{\mathrm{L}}^\theta F = \dim_{\mathrm{L}}^\theta F$, as required.

  To see that (ii) implies (iii), write $\varphi(\theta) = (1 - \theta)\dim_{\mathrm{L}}^\theta F$ so $\varphi \in \mathcal{H}(d)$. Then, observe for all $0 < \theta \leq 1$ that

$$(1 - \theta)\inf_{0<\lambda\leq\theta}\frac{\varphi(\lambda)}{1 - \lambda} = \inf_{0<\lambda\leq\theta}\phi_{\alpha,\lambda,\varphi(\lambda)}(\theta) = \inf_{0<\lambda\leq1}\phi_{\alpha,\lambda,\varphi(\lambda)}(\theta).$$

But by (3.2), the function on the left is precisely $(1 - \theta)\underline{\dim}_{\mathrm{L}}^\theta F$.

  Finally, (iii) implies (i) since $\phi_{d,\lambda,t} \in \mathcal{M}(d)$ and it is easy to check that $\mathcal{M}(d)$ is closed under pointwise infima (for instance, see the argument in the proof of Proposition 2.4). □
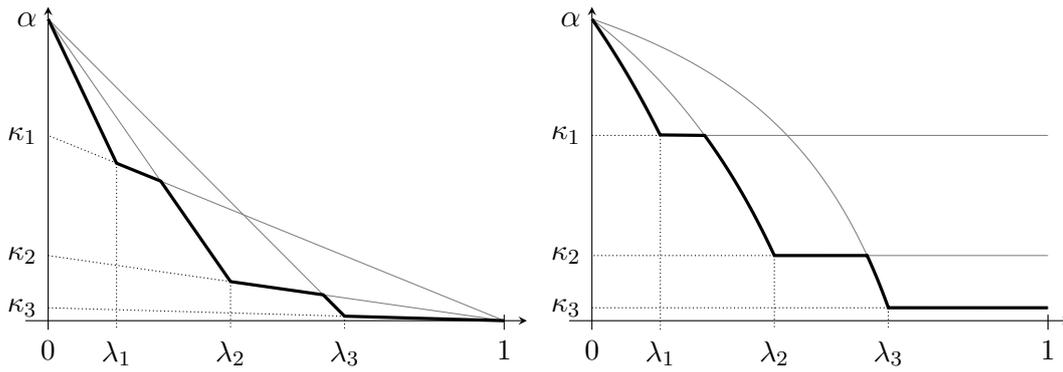
(A) The functions $\phi_{\alpha,\lambda_i,t_i}$ (B) The functions $\theta \mapsto \phi_{\alpha,\lambda_i,t_i}(\theta)/(1-\theta)$

FIGURE 2. A function obtained as the minimum of three functions $\phi_{\alpha,\lambda_i,t_i}$, defined in §3.4, and the corresponding lower spectrum. Here, $\kappa_i = t_i/(1-\lambda_i)$.

3.5. Two examples of lower spectra. We consider two explicit families of examples of lower spectra to highlight some of the possible behaviour.

3.5.1. Convex functions. Suppose $f\colon [0,1] \to [0,\alpha]$ is decreasing and convex with $f(0) \leq \alpha$ and $f(1) = 0$. For $0 < \theta \leq 1$, applying convexity on the interval $[0,\theta]$, we see that f satisfies (W), and applying convexity on the interval $[\theta,1]$, we see that f satisfies (M). Therefore $f \in \mathcal{M}(\alpha)$. This shows, for example, that the Lipschitz property proven in Lemma 3.3 is essentially optimal, and demonstrates that there are many sets $F \subset \mathbb{R}^d$ for which $\theta \mapsto \phi(\theta)$ and $\theta \mapsto \dim_{\mathrm{L}}^\theta F$ are not Lipschitz on $(0,1)$.

However, not all functions in $\mathcal{M}(\alpha)$ are of this form. One can construct non-convex examples by taking an infimum of functions of the form $\phi_{\alpha,\lambda,t}$ from §3.4. See Figure 2 for a depiction.

3.5.2. Non-monotonic lower spectrum. In this section, we give a simple example of a set with non-monotonic lower spectrum, different to the example from [FY18b, §8]. The construction is similar in spirit to the family of examples in [Rut24, §3.3.2].

The example will be an element of $\mathcal{H}(\alpha)$ for $\alpha \geq 0$, and it is parametrized by numbers $a_1, a_2 \in (0,1]$ and a slope $0 \leq \kappa \leq \alpha$. Write $\boldsymbol{a} = (a_1, a_2, \kappa)$. The function $q_{\boldsymbol{a}}$ is defined so that it is continuous, has slope $-\kappa$ on the intervals $[a_1 a_2, a_1]$ and $[a_2, 1]$, and satisfies inequality (W) with equality on the intervals $[0, a_1 a_2]$ and $[a_1, a_2]$. The precise definition is a bit tedious to state. Set

$$\alpha_1 = \frac{\alpha - \kappa(a_1 - a_1 a_2 + 1 - a_2) + \alpha_2(a_2 - a_1)}{a_1}$$
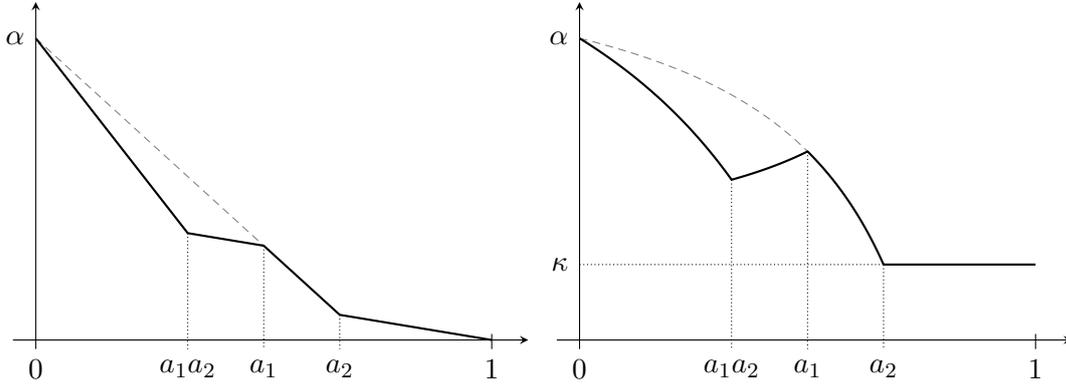
$$\alpha_2 = \frac{\alpha - \kappa(1 - a_2)}{a_2}$$

(A) Plot of q_a.   (B) Plot of $\theta \mapsto q_a(\theta)/(1-\theta)$.

FIGURE 3. A plot of the function q_a where $\alpha = 1$ and $\boldsymbol{a} = (1/2, 2/3, 1/4)$, as well as the corresponding lower spectrum.

and then set

$$
q_{\boldsymbol{a}}(\theta) = \begin{cases}
\alpha_1(a_1 a_2 - \theta) + \kappa(a_1 - a_1 a_2 + 1 - a_2) + \alpha_2(a_2 - a_1) & : 0 \le \theta \le a_1 \\
\kappa(a_1 - \theta + 1 - a_2) + \alpha_2(a_2 - a_1) & : a_1 a_2 \le \theta \le a_2 \\
\kappa(1 - a_2) + \alpha_2(a_2 - \theta) & : a_1 \le \theta \le a_2 \\
\kappa(1 - \theta) & : a_2 \le \theta \le 1
\end{cases}
$$

See Figure 3 for a depiction of the function $q_{\boldsymbol{a}}$. Note that $q_{\boldsymbol{a}}$ has slope $-\alpha_1$ on $[0, a_1 a_2]$ and slope $-\alpha_2$ on $[a_1, a_2]$. In general, $\kappa \le \alpha_2 \le \alpha_1$, and all of the inequalities are strict if $\kappa < \alpha$. Also, it is easy to see that $q_{\boldsymbol{a}}$ satisfies (W).

In order to prove (S), fix $\theta \in (0, 1]$ and consider the function

$$
\phi_{\boldsymbol{a},\theta}(\lambda) = \frac{q_{\boldsymbol{a}}(\theta\lambda) - q_{\boldsymbol{a}}(\theta)}{\theta}.
$$

We must show that $\phi_{\boldsymbol{a},\theta} \ge q_{\boldsymbol{a}}$ for all $\theta \in (0, 1]$.

To prove this, observe that $\phi_{\boldsymbol{a},\theta}$ acts as follows. For $0 < \theta \le 1$, let

$$
\begin{aligned}
I_1(\theta) &= [0, a_1 a_2/\theta] \cap (0, 1], \\
I_2(\theta) &= [a_1 a_2/\theta, a_1/\theta] \cap (0, 1], \\
I_3(\theta) &= [a_1/\theta, a_2/\theta] \cap (0, 1], \\
I_4(\theta) &= [a_2/\theta, 1] \cap (0, 1].
\end{aligned}
$$

Of course, some of the intervals $I_j(\theta)$ will be empty for small θ. For an interval I, write $|I|$ to denote its length. Then $\phi_{\boldsymbol{a},\theta}$ is the continuous function with slopes $-\alpha_1$ on $I_1(\theta)$, $-\kappa$ on $I_2(\theta)$, $-\alpha_2$ on $I_3(\theta)$, and $-\kappa$ on $I_4(\theta)$ such that $\phi_{\boldsymbol{a},\theta}(1) = 0$.

Recall that $\kappa \le \alpha_2 \le \alpha_1$. We consider four cases depending on the value of $\theta \in (0, 1]$:

1. Suppose $a_2 < \theta$. Then $|I_j(\theta)| \ge |I_j(1)|$ for $j \le 3$ and $|I_4(\theta)| \le |I_4(1)|$ so that $q_{\boldsymbol{a}} \le \phi_{\boldsymbol{a},\theta}$.

2. Suppose $a_1 < \theta \leq a_2$. Then $I_4(\theta) = \varnothing$, $|I_2(\theta)| \leq |I_4(1)|$, and $a_1 \leq a_1a_2/\theta < a_2$. Therefore, $q_{\boldsymbol{a}}(a_1a_2/\theta) \leq \phi_{\boldsymbol{a},\theta}(a_1a_2/\theta)$ and it follows that $q_{\boldsymbol{a}} \leq \phi_{\boldsymbol{a},\theta}$.

3. Suppose $a_1a_2 \leq \theta < a_1$. Then $I_3(\theta) = I_4(\theta) = \varnothing$ and $|I_2(\theta)| \leq |I_4(1)|$, so $q_{\boldsymbol{a}} \leq \phi_{\boldsymbol{a},\theta}$.

4. Suppose $\theta < a_2a_2$. Then $q_{\boldsymbol{a}}$ has constant slope $-\alpha_1$ and the result is trivial.

Therefore $q_{\boldsymbol{a}}$ satisfies (S), so that $q_{\boldsymbol{a}} \in \mathcal{H}(\alpha)$.

4. Uniform sets and lower spectra

We conclude this paper with our results concerning uniform sets. This includes the construction of uniform sets used in the proof of Theorem B, as well as the proof of Theorem C.

4.1. Uniform sets. First, we recall the definition of the (upper) two-scale branching function from [OR25+]. For a metric space X we let $\beta = \beta_X \colon \Delta \to [0, \infty]$ denote the function

$$\beta_X(u, v) = \log \sup_{x \in X} N_{2^{-u}}(B(x, 2^{-v})).$$

This is the upper variant of the branching function ζ_X. The following properties of β_X are easy to check:
  (i) $\beta_X(u, u) = 0$ for all $u \in \mathbb{R}$.
 (ii) $\beta_X(u, v)$ is increasing in u and decreasing in v.
(iii) For all $v \leq w \leq u$,

$$\beta_X(u, v) \leq \beta_X(u, w) + \beta_X(w, v).$$

For a general metric space X the inequality $\zeta_X \leq \beta_X$ always holds, and clearly these functions can be very different. For much more detail concerning the function β_X, we refer the reader to [OR25+].

Definition 4.1. Let $\eta \colon \mathbb{R} \to \mathbb{R}$. We say that a metric space X is *η-uniform* if $\lim_{u \to \infty} u^{-1}\eta(u) = 0$ and

$$\beta_X(u, v) \leq \zeta_X(u, v) + \eta(u) \quad \text{for all} \quad (u, v) \in \Delta.$$

We say that X is *uniform* if it is η-uniform for some η.

As long as the space X is somewhat nice (so that covers and packings are comparable), the two-scale branching functions of uniform sets are determined entirely by covering numbers.

Definition 4.2. For $\alpha \geq 0$, we let $\mathcal{Y}(\alpha)$ denote the set of increasing α-Lipschitz functions $f \colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$.

We now obtain our description of uniform sets.

Corollary 4.3. *Let $d \in \mathbb{N}$. Then there is a constant $C_d > 0$ so that for any η-uniform subset $X \subset \mathbb{R}^d$, there is a function $f \in \mathcal{Y}(d)$ such that for all $0 \leq v \leq u$,*

$$|\beta_X(u, v) - (f(u) - f(v))| \leq \eta(u) + C_d,$$
$$|\zeta_X(u, v) - (f(u) - f(v))| \leq \eta(u) + C_d.$$

Proof. Let $X \subset \mathbb{R}^d$ be η-uniform and set

$$g(u) = \beta_X(u, 0).$$

In $\mathbb{R}^d$, there is a constant c_d so that for all $z \in \mathbb{R}$,

$$(4.1) \qquad\qquad 0 \leq \beta_X(z, 0) - \zeta_X(z, 0) \leq c_d.$$

Now, by subadditivity followed by η-uniformity,

$$\beta_X(u, 0) - \beta_X(v, 0) \leq \beta_X(u, v) \leq \zeta_X(u, v) + \eta(u).$$

Similarly, by superadditivity followed by η-uniformity,

$$\zeta_X(u, 0) - \zeta_X(v, 0) \geq \zeta_X(u, v) \geq \beta_X(u, v) - \eta(u).$$

But by (4.1),

$$|(\beta_X(u, 0) - \beta_X(v, 0)) - (\zeta_X(u, 0) - \zeta_X(v, 0))| \leq 2c_d$$

Thus the conclusion holds with $C_d = 2c_d$ and g in place of f.

To finish the proof, we note that g is increasing, $g(0) = 0$, and for all $u \in \mathbb{R}$ and $z \geq 0$,

$$g(u + z) \leq g(u) + dz + c_d$$

for some constant $c_d \geq 0$. By taking the supremum over d-Lipschitz functions bounded above by g and then adding a fixed constant, we get $f \in \mathcal{Y}(d)$ such that $g(u) - c_d \leq f \leq g(u) + c_d$ for all $u \geq 0$. This completes the proof. $\qquad\square$

Remark 4.4. Setting $v = 0$, we see that the function f provided by Corollary 4.3 satisfies

$$\beta_X(u, 0) = f(u) + o(u).$$

(This is also clear from the proof itself.) For example, if $\operatorname{diam} X \leq 1$, then $\beta_X(u, 0) = \log N_{2^{-u}}(X)$ is just the logarithm of the covering number at scale 2^{-u}. Corollary 4.3 says that these covering numbers approximately determine all of the two-scale covering numbers of a uniform set X.

4.2. Constructing uniform sets. A standard way to construct a uniform set in $\mathbb{R}^d$ is as follows. Suppose $f \in \mathcal{Y}(d)$ is arbitrary. By taking a supremum over all integer-valued d-Lipschitz functions with $f(0) = 0$ bounded above by f, there exists a function $h \colon \mathbb{Z} \cap [0, \infty) \to \mathbb{Z} \cap [0, \infty)$ which is also an increasing and d-Lipschitz function with $h(0) = 0$, such that

$$f(k) - 1 < h(k) \le f(k) \quad \text{for all} \quad k \ge 0.$$

Then, define a sequence $(a_k)_{k=1}^{\infty} \in \{0, \ldots, d\}^{\mathbb{N}}$ by the rule $a_k = h(k) - h(k-1)$. We inductively construct nested families of dyadic cubes as follows. Let $\mathcal{Q}_0 = \{[0, 1]^d\}$. Now, suppose we have constructed $\mathcal{Q}_k$ as a union of dyadic cubes at level k, for some $k \ge 0$. Then, replace each cube $Q \in \mathcal{Q}_k$ with $2^{a_{k+1}}$ sub-cubes at level $k + 1$ (the precise choice of sub-cube is irrelevant). Finally, set

$$K_f = \bigcap_{k=0}^{\infty} \bigcup_{Q \in \mathcal{Q}_k} Q.$$

Since each level k dyadic cube contains exactly $2^{h(n) - h(k)}$ dyadic cubes at level n for $k \le n$, it follows that K_f is an $O_d(1)$-uniform set with branching function

$$\zeta_{K_f}(u, v) = f(u) - f(v) + O_d(1).$$

Moreover, if $f(u) = 0$ for some $u \ge 0$, then K_f is contained in a single dyadic cube of side-length $2^{-\lceil u \rceil}$.

To summarize, we have proven the following.

Proposition 4.5. *Let $d \in \mathbb{N}$ and $f \in \mathcal{Y}(d)$. Suppose $f(z) = 0$ for some $z \ge 0$. Then there exists a compact $O_d(1)$-uniform set $K_f \subset [0, 2^{-z}]^d$ such that*

$$\zeta_{K_f}(u, v) = f(u) - f(v) + O_d(1)$$

for all $0 \le v \le u$.

4.3. Classification of lower spectra of uniform sets. The goal of this section is to study the lower spectra of uniform sets and to prove Theorem C. We begin with the space of functions.

Definition 4.6. Let $\alpha \ge 0$. We let $\mathcal{U}(\alpha)$ denote the functions $\varphi \colon [0, 1] \to [0, \alpha]$ which are α-Lipschitz and satisfy (S).

Comparing the definition of $\mathcal{U}(\alpha)$ with $\mathcal{H}(\alpha)$, we see that we have replaced the convexity property (W) with the α-Lipschitz property. The α-Lipschitz property is strictly stronger.

Lemma 4.7. *For all $\alpha \ge 0$, we have $\mathcal{U}(\alpha) \subset \mathcal{H}(\alpha)$, and the inclusion is strict if $\alpha > 0$.*

Proof. Let $\varphi \in \mathcal{U}(\alpha)$ and recall that $\varphi(1) = 0$. Since φ is α-Lipschitz, $\alpha\lambda \le \alpha - \varphi(\lambda)$ and therefore

$$\varphi(\lambda\theta) - \varphi(\lambda) \le \alpha\lambda(1 - \theta) \le (1 - \theta)(\alpha - \varphi(\lambda)).$$

This proves (W), and therefore $\varphi \in \mathcal{H}(\alpha)$.

To see that the inclusion is strict when $\alpha > 0$, use for example the functions $\phi_{\alpha,\lambda,t} \in \mathcal{H}(\alpha)$ defined in §3.4. $\qquad\square$

We now prove that $\mathcal{U}(d)$ is precisely the set of lower spectra of uniform subsets of $\mathbb{R}^d$.

Restatement (of Theorem C). *Let $d \in \mathbb{N}$ and $\varphi \colon [0,1] \to [0,d]$. Then there is a uniform set $F \subset \mathbb{R}^d$ with $(1-\theta)\dim_{\mathrm{L}}^\theta F = \varphi(\theta)$ for all $\theta \in (0,1]$ if and only if $\varphi \in \mathcal{U}(d)$.*

Proof. First suppose that $F \subset \mathbb{R}^d$ is η-uniform and write $\varphi(\theta) = (1-\theta)\dim_{\mathrm{L}}^\theta F$. By Corollary 4.3, get $f \in \mathcal{Y}(d)$ and a constant $C_d \geq 0$ such that for all $0 \leq v \leq u$,

$$|\zeta_F(u,v) - (f(u) - f(v))| \leq C_d + \eta(u).$$

Thus if we define $g(u,v) = f(u) - f(v)$, certainly $g \in \mathcal{L}(d)$. In particular, $\Lambda(\zeta_F) = \Lambda(g)$ by Lemma 3.6 so by Corollary 3.7, for $0 < \theta \leq 1$,

$$\varphi(\theta) = \liminf_{u\to\infty} \frac{f(u) - f(\theta u)}{u}.$$

Since f is d-Lipschitz, $\theta \mapsto u^{-1}(f(u) - f(\theta u))$ is d-Lipschitz, so φ is d-Lipschitz and therefore $\varphi \in \mathcal{U}(d)$.

For the converse direction, for a given $f \in \mathcal{Y}(d)$, the corresponding uniform set $K_f \subset [0,1]^d$ provided by Proposition 4.5 satisfies

$$(4.2) \qquad (1-\theta)\dim_{\mathrm{L}}^\theta K_f = \liminf_{u\to\infty} \frac{f(u) - f(\theta u)}{u}.$$

Therefore it suffices to choose $f \in \mathcal{Y}(d)$ such that this limit infimum is exactly the function $\varphi \in \mathcal{U}(d)$. Let $0 = v_1 < u_1 < v_2 < u_2 < \cdots$ be a sequence such that

$$\lim_{k\to\infty} \frac{v_k}{u_k} = \lim_{k\to\infty} \frac{u_k}{v_{k+1}} = 0.$$

Begin with $f(0) = 0$. Now, suppose we have defined f on the interval $[0, v_k]$ for some $k \in \mathbb{N}$. Let $\theta_k = v_k/u_k$. With (4.2) in mind, we would like to define f on the interval $[v_k, u_k]$ such that, for $\theta_k \leq \theta \leq 1$,

$$(4.3) \qquad f(u_k) - f(\theta u_k) = u_k \varphi(\theta).$$

Recalling that the value $f(v_k)$ was defined in the previous stage of the construction, for $v_k \leq u \leq u_k$ we define

$$f(u) = f(v_k) + u_k \varphi(\theta_k) - u_k \varphi(u/u_k)$$

and it is easy to check that (4.3) is satisfied since $\varphi(1) = 0$. Since φ is decreasing and d-Lipschitz, f is increasing and d-Lipschitz on $[0, u_k]$. Finally, for $u_k \leq u \leq v_{k+1}$, we simply define

$$(4.4) \qquad f(u) = d(u - u_k) + f(u_k).$$

Therefore, $f \in \mathcal{Y}(d)$.

It remains to show that

$$\liminf_{u \to \infty} \frac{f(u) - f(\theta u)}{u} = \varphi(\theta)$$

for all $0 < \theta \leq 1$. Certainly

$$\liminf_{u \to \infty} \frac{f(u) - f(\theta u)}{u} \leq \lim_{k \to \infty} \frac{f(u_k) - f(\theta u_k)}{u_k} = \varphi(\theta)$$

since the final equality holds for all k sufficiently large by (4.3). To complete the proof, fix $0 < \kappa \leq 1$: it suffices to show that for all u sufficiently large,

(4.5) $$f(u) - f(\kappa u) \geq u\varphi(\kappa).$$

Let N be sufficiently large so that for all $k \geq N$, $v_k/u_k \leq \kappa$ and $u_k/v_{k+1} \leq \kappa$, and let $u \geq v_N/\kappa$ be arbitrary. The choice of N implies that there are four cases depending on the values of u and κu:

1. *Suppose $v_k \leq \kappa u \leq u \leq u_k$ for some $k \in \mathbb{N}$. Let $u = \theta u_k$ so that $\kappa u = \kappa \theta u_k$.* Then by (S),

$$f(u) - f(\kappa u) = u_k \varphi(\kappa \theta) - u_k \varphi(\theta) \geq u_k \theta \varphi(\kappa) = u\varphi(\kappa).$$

2. *Suppose $u_k \leq \kappa u \leq u \leq u_{k+1}$ for some $k \in \mathbb{N}$. Since φ is d-Lipschitz, $\varphi(\kappa) \leq d(1 - \kappa)$* so

$$f(u) - f(\kappa u) = u(1 - \kappa)d \geq u\varphi(\kappa).$$

3. *Suppose $v_k \leq \kappa u \leq u_k \leq u \leq v_{k+1}$ for some $k \in \mathbb{N}$. Let λ be such that $\kappa u = \lambda u_k$ and write $\theta = \kappa/\lambda \leq 1$.* Then by (W) (recalling that $\mathcal{U}(d) \subset \mathcal{H}(d)$),

$$\begin{aligned} f(u) - f(\kappa u) &= \alpha(u - u_k) + u_k \varphi(\lambda) \\ &= u(\alpha(1 - \theta) + \theta\varphi(\lambda)) \\ &\geq u\varphi(\lambda\theta) = u\varphi(\kappa). \end{aligned}$$

4. *Suppose $u_{k-1} \leq \kappa u \leq v_k \leq u \leq u_k$ for some $k \in \mathbb{N}$. Let λ be such that $\lambda u = v_k$.* Applying case 1 on the interval $[\lambda u, u]$ and then using the fact that φ is α-Lipschitz,

$$\begin{aligned} f(u) - f(\kappa u) &= f(u) - f(\lambda u) + \alpha(\lambda u - \kappa u) \\ &\geq u(\varphi(\lambda) + \alpha(\lambda - \kappa)) \\ &\geq u\varphi(\kappa). \end{aligned}$$

Thus (4.5) follows, completing the proof. $\square$

To conclude, we note that one can prove similar results as proven in §3.4 for the monotone lower spectrum of uniform sets. Given $\alpha \geq 0$, $\lambda \in (0, 1]$, and $0 \leq t \leq \alpha(1 - \lambda)$, we define the function

$$\psi_{\alpha,\lambda,t}(\theta) = \begin{cases} t + \alpha(\theta - \lambda) & : 0 < \theta \leq \lambda, \\ t\frac{1-\theta}{1-\lambda} & : \lambda \leq \theta \leq 1. \end{cases}$$

This is the function which has constant slope $-\alpha$ on the interval $[0, \lambda]$, with value t at λ, and constant slope on $[\lambda, 1]$ with value 0 at 1. The condition on t ensures that $\psi_{\alpha,\lambda,t} \in \mathcal{M}(\alpha) \cap \mathcal{U}(\alpha)$. Recalling the definition of ϕ from §3.4, we note that $\psi_{\alpha,\lambda,t} \leq \phi_{\alpha,\lambda,t}$, and moreover the inequality is strict on $(0, \lambda)$ if and only $t < \alpha(1-\lambda)$. Following an identical approach to that in §3.4, one can obtain the following result.

Proposition 4.8. *Let $d \in \mathbb{N}$ and $\varphi \colon (0, 1] \to [0, d]$. Then the following are equivalent.*
  (i) φ is α-Lipschitz and satisfies (M).
 (ii) There exists a uniform $F \subset \mathbb{R}^d$ such that $(1 - \theta)\underline{\dim}_{\mathrm{L}}^{\theta} F = \varphi(\theta)$ for all $\theta \in (0, 1]$.
(iii) φ is an infimum of a family of functions of the form $\psi_{d,\lambda,t}$ for $\lambda \in (0, 1]$ and $0 \leq t \leq \alpha(1 - \lambda)$.

A<small>CKNOWLEDGEMENTS</small>

A. M<small>ONOTONIZATION OF THE LOWER SPECTRUM</small>

The following result was proven under the additional assumption that X is doubling and uniformly perfect in [CWC20, Theorem 1.1], and then with only the doubling assumption in [HT21, Theorem A.2]. Here we dispense of the doubling assumption, and moreover give a simple proof which only uses monotonicity of covering numbers.

Theorem A.1. *Let X be a non-empty metric space. Then for all $0 < \theta < 1$,*

$$\underline{\dim}_{\mathrm{L}}^{\theta} X = \inf_{0 < \lambda \leq \theta} \dim_{\mathrm{L}}^{\lambda} X.$$

Proof. It suffices to prove that

(A.1) $$\underline{\dim}_{\mathrm{L}}^{\theta} X \geq \inf_{0 < \lambda \leq \theta} \dim_{\mathrm{L}}^{\lambda} X$$

since the opposite inequality is immediate from the definition.

Let γ be defined as in the proof of Corollary 3.7; that is, for $v \leq u$,

$$\gamma(u, v) = \log \inf_{x \in X} N_{2^{-u}}(B(x, 2^{-v})).$$

The only facts about γ that we will require are that γ is non-negative, $v \mapsto \gamma(u, v)$ is decreasing for all u, and (from the proof of Corollary 3.7) for $0 < \theta < 1$,

$$\dim_{\mathrm{L}}^{\theta} X = \liminf_{u \to \infty} \frac{\gamma(u, \theta u)}{u(1 - \theta)}.$$

We may assume that $\underline{\dim}_{\mathrm{L}}^{\theta} X < \infty$, or else the inequality is trivial. Let $t > s > \underline{\dim}_{\mathrm{L}}^{\theta} X$ be arbitrary. By definition of $\underline{\dim}_{\mathrm{L}}^{\theta} X$, for all $n \in \mathbb{N}$ there is a $u_n \geq 0$ and $\lambda_n \in [0, \theta]$ such that

$$\gamma(u_n, \lambda_n u_n) \leq s u_n (1 - \lambda_n) - n.$$

Since γ is non-negative, we must have $\lim_{n \to \infty} u_n = \infty$. Passing to a subsequence if necessary, we may assume that $\lim_{n \to \infty} \lambda_n = \lambda \in [0, \theta]$. If $\lambda = \theta$, then $\lambda_n \leq \theta$ for all $n \in \mathbb{N}$ so that

$$\gamma(u_n, \theta u_n) \leq \gamma(u_n, \lambda_n u_n) \leq s u_n (1 - \lambda_n).$$

Since $\theta < 1$, $\lim_{n \to \infty} (1 - \lambda_n)^{-1} = (1 - \theta)^{-1}$ and therefore

$$\dim_{\mathrm{L}}^{\theta} X \leq \liminf_{n \to \infty} \frac{\gamma(u_n, \theta u_n)}{u_n(1 - \lambda_n)} \leq s < t.$$

Otherwise, $\lambda < \theta$, and let $\lambda < \kappa \leq \theta$ be such that $s(1 - \lambda)/(1 - \kappa) < t$. Since $\lambda_n \leq \kappa$ for all n sufficiently large,

$$\gamma(u_n, \kappa u_n) \leq \gamma(u_n, \lambda_n u_n) \leq s u_n (1 - \lambda_n).$$

Therefore, exactly as above,

$$\dim_{\mathrm{L}}^{\kappa} X \leq \liminf_{n \to \infty} \frac{\gamma(u_n, \lambda_n u_n)}{u_n(1 - \lambda_n)} \cdot \frac{1 - \lambda_n}{1 - \kappa} \leq s \frac{1 - \lambda}{1 - \kappa} \leq t.$$

In any case, we have shown that for all $t > \underline{\dim}_{\mathrm{L}}^{\theta} X$ there is a $\lambda \in (0, \theta]$ so that $\dim_{\mathrm{L}}^{\lambda} X \leq t$. Therefore (A.1) follows. $\square$

B.   SUMMARY OF THE MAIN INEQUALITIES

In this appendix, we list all of the inequalities used in the various classification theorems and state the key properties along with short proofs. More detailed versions of these proofs can be found throughout the text in the places where the statement is required.

Definition B.1. Let $\alpha \geq 0$ and let $\varphi\colon (0,1] \to [0,\alpha]$ be a function. We consider the following inequalities concerning φ.

(S) *Superadditive:* For all $\lambda, \theta \in (0,1]$,

$$\varphi(\lambda\theta) \geq \varphi(\theta) + \theta\varphi(\lambda).$$

(W) *Weak α-Lipschitz:* For all $\lambda, \theta \in (0,1]$,

$$\varphi(\lambda\theta) \leq (1-\theta)\alpha + \theta\varphi(\lambda).$$

(M) *Monotone:* The function $\theta \mapsto \varphi(\theta)/(1-\theta)$ is decreasing.

(L) *α-Lipschitz:* The function φ is α-Lipschitz.

We note from the introduction and Proposition 4.8 that the inequalities relevant for the various classification results are as follows:

- $\theta \mapsto \dim_{\mathrm{L}}^{\theta} F$ (for general F) uses (S) and (W)
- $\theta \mapsto \dim_{\mathrm{L}}^{\theta} F$ (for uniform F) uses (S) and (L)
- $\theta \mapsto \underline{\dim}_{\mathrm{L}}^{\theta} F$ (for general F) uses (M) and (W)
- $\theta \mapsto \underline{\dim}_{\mathrm{L}}^{\theta} F$ (for uniform F) uses (M) and (L)

Let us make the following simple observations.

1. (S) implies that φ is decreasing, since $\theta\varphi(\lambda) \geq 0$. In particular, we may define $\varphi(0) = \lim_{\theta \to 0} \varphi(\theta)$. Observe that all of the above inequalities are satisfied for all $\lambda, \theta \in [0,1]$ with this definition.

2. (S) implies that $\varphi(1) = 0$, by taking $\lambda = \theta = 1$.

3. (W) implies that $\varphi(\theta) \leq \alpha(1-\theta)$, by taking $\lambda = 1$ and recalling 2. In particular, (W) also implies that $\varphi(1) = 0$.

4. (W) is a weak type of convexity at 0: for all $\lambda \in [0,1]$, the graph of φ lies below the line segment connecting $(0, \alpha)$ and $(\lambda, \varphi(\lambda))$.

5. Using 4, it follows that (W) holds if and only if the function

$$\theta \mapsto \frac{\alpha - \varphi(\theta)}{\theta}$$

is decreasing. In particular, φ is $\theta^{-1}(\alpha - \varphi(\theta))$-Lipschitz on $[\theta, 1]$.

6. Rearranging (S), for all $\lambda, \theta \in (0,1]$,

$$\frac{\varphi(\lambda\theta) - \varphi(\theta)}{\theta - \lambda\theta} \geq \frac{\varphi(\lambda)}{1-\lambda}.$$

The left hand side is the negative of slope of the line segment from $(\lambda\theta, \varphi(\lambda\theta))$ to $(\theta, \varphi(\theta))$, and the right hand side is the negative slope of the line segment from $(\lambda, \varphi(\lambda))$ to $(1,0)$.

7. (M) has a geometric interpretation: for any $0 < \theta \leq 1$, φ is bounded above on $[\theta, 1]$ by the line segment joining $(\theta, \varphi(\theta))$ and $(1,0)$. In particular, it follows from 6 that (M) implies (S).

8. If (L) holds and $\varphi(1) = 0$, then (W) holds. Indeed, since φ is α-Lipschitz, $\alpha\lambda \leq \alpha - \varphi(\lambda)$ and therefore

$$\varphi(\lambda\theta) - \varphi(\lambda) \leq \alpha\lambda(1 - \theta) \leq (1 - \theta)(\alpha - \varphi(\lambda)).$$

Rearranging this inequality yields (W).

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