Tangents and pointwise Assouad dimension of invariant sets

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ABSTRACT. We study the fine scaling properties of sets satisfying various weak forms of invariance. Our focus is on the interrelated concepts of (weak) tangents, Assouad dimension, and a new localized variant which we call the pointwise Assouad dimension. For general attractors of possibly overlapping bi-Lipschitz iterated function systems, we establish that the Assouad dimension is given by the Hausdorff dimension of a tangent at some point in the attractor. Under the additional assumption of self-conformality, we moreover prove that this property holds for a subset of full Hausdorff dimension. We then turn our attention to an intermediate class of sets: namely planar self-affine carpets. For Gatzouras–Lalley carpets, we obtain precise information about tangents which, in particular, shows that points with large tangents are very abundant. However, already for Barański carpets, we see that more complex behaviour is possible.

CONTENTS

1. INTRODUCTION

One of the most fundamental concepts at the intersection of analysis and geometry is the notion of a *tangent*. For sets exhibiting a high degree of local regularity—such as manifolds, or rectifiable sets—at almost every point in the set and at all sufficiently high resolutions, the set looks essentially linear. Moreover, the concept of a tangent is particularly relevant in the study of a much broader class of sets: those equipped with some form of dynamical invariance. This relationship originates in the pioneering work of Furstenberg, where one associates to a set a certain dynamical system of "zooming in". Especially in the past two decades, the study of tangent measures has played an important role in the resolution of a number of long-standing problems concerning sets which look essentially the same at all small scales; see, for example, [HS12; HS15; KSS15; Shm19; Wu19].

In contrast, (weak) tangents also play an important role in the geometry of metric spaces. One of the main dimensional quantities in the context of embeddability properties of metric spaces is the Assouad dimension, first introduced in [Ass77]. It turns out that the Assouad dimension, which bounds the worst-case scaling at all locations and all small scales, is precisely the maximal Hausdorff dimension of weak tangents, i.e. sets which are given as a limit of small pieces of enlarged copies of the original set; see [KOR17]. We refer the reader to the books [Fra20; MT10; Rob11] for more background and context on the importance of Assouad dimension in a variety of diverse applications.

In this document, we study the interrelated concepts of tangents and Assouad dimension, with an emphasis on sets with a weak form of dynamical invariance. Our motivating examples include attractors of iterated function systems where the maps are affinities (or even more generally bi-Lipschitz contractions); or the maps are conformal and there are substantial overlaps. In both of these situations, the sets exhibit a large amount of local inhomogeneity. As we will see, these classes of sets exhibits a rich variety of behaviour while still retaining some fundamental properties.

(A) Gatzouras–Lalley (B) Barański

FIGURE 1. Some self-embeddable sets, which are attractors of the iterated function systems depicted in Figure 2.

1.1. Weak tangents, tangents, and pointwise Assouad dimension. Throughout, we will work in $\mathbb{R}^d$ for some $d \in \mathbb{N}$, though many of our results hold in the broader context of bounded doubling metric spaces. We let $B(x, r)$ denote the closed ball with centre x and radius r.

Now, fix a compact set $K \subset \mathbb{R}^d$. We say that a compact set $F \subset B(0, 1)$ is a *weak tangent* of $K \subset \mathbb{R}^d$ if there exists a sequence of similarity maps $(T_k)_{k=1}^{\infty}$ with similarity ratios λ_k diverging to infinity such that $0 \in T_k(K)$ and

$$F = \lim_{k\to\infty} T_k(K) \cap B(0, 1)$$

with respect to the Hausdorff metric on compact subsets of $B(0, 1)$. We denote the set of weak tangents of K by $\operatorname{Tan}(K)$. More strongly, we say that F is a *tangent of K at x* if F is a weak tangent and the similarity maps T_k are homotheties which map x to 0; i.e. $T_k(y) = \lambda_k(y - x)$. We denote the set of tangents of K at x by $\operatorname{Tan}(K, x)$. We refer the reader to §2.1 for precise definitions.

Closely related to the notion of a weak tangent is the *Assouad dimension* of K, which is the dimensional quantity

$$\dim_{\mathrm{A}} K = \inf\Big\{s : \exists C > 0\, \forall 0 < r \le R < 1\, \forall x \in K \\ N_r(B(x, R) \cap K) \le C\Big(\frac{R}{r}\Big)^s\Big\}.$$

Here, for a general bounded set F, $N_r(F)$ is the smallest number of closed balls with radius r required to cover F. It always holds that $\dim_{\mathrm{H}} K \le \overline{\dim}_{\mathrm{B}} K \le \dim_{\mathrm{A}} K$, where $\dim_{\mathrm{H}} K$ and $\overline{\dim}_{\mathrm{B}} K$ denote the Hausdorff and upper box dimensions respectively. In some sense, the Assouad dimension is the largest reasonable notion of dimension which can be defined using covers. Continuing the analogy with tangents, we also introduce a localized version of the Assouad dimension which we call the *pointwise Assouad dimension*. Given $x \in K$, we set

$$\dim_{\mathrm{A}}(K, x) = \inf\Big\{s : \exists C > 0\, \exists \rho > 0\, \forall 0 < r \le R < \rho \\ N_r(B(x, R) \cap K) \le C\Big(\frac{R}{r}\Big)^s\Big\}.$$

The choice of $\rho > 0$ in the definition of $\dim_A(K, x)$ ensures a sensible form of bi-Lipschitz invariance: if $f\colon K \to K'$ is bi-Lipschitz, then $\dim_A(K, x) = \dim_A(f(K), f(x))$. It is immediate from the definition that

$$\dim_A(K, x) \leq \dim_A K.$$

Moreover, if for instance K is Ahlfors–David regular, then $\dim_A(K, x) = \dim_A K$ for all $x \in K$. We note here that an analogous notion of pointwise Assouad dimension for measures was introduced recently in [Ant22+].

An observation which goes back essentially to Furstenberg, but was observed explicitly in [KOR17], is that the Assouad dimension is characterized by weak tangents:

$$\dim_A K = \max\{\dim_H F : F \in \operatorname{Tan}(K)\}.$$

Motivated by this relationship, our primary goal in this document is to address the following questions:

- Does it hold that $\dim_A(K, x) = \max\{\dim_H F : F \in \operatorname{Tan}(K, x)\}$?
- Is there necessarily an $x_0 \in K$ so that $\dim_A K = \dim_H F$ for some $F \in \operatorname{Tan}(K, x_0)$? If not, is there an $x_0 \in K$ so that $\dim_A K = \dim_A(K, x_0)$?
- What is the structure of the level set of pointwise Assouad dimension $\{x \in K : \dim_A(K, x) = \alpha\}$ for some $\alpha \geq 0$?

In the following section, we discuss our main results and provide some preliminary answers which indicate that answers to these questions are, in general, quite subtle.

1.2. Main results and outline of paper. We begin by stating some easy properties of the pointwise Assouad dimension for general compact sets $K \subset \mathbb{R}^d$. Firstly, by Proposition 2.2,

$$\sup\{\overline{\dim}_B F : F \in \operatorname{Tan}(K, x)\} \leq \dim_A(K, x) \leq \dim_A K$$

for all $x \in K$ and, by Proposition 2.8 (ii), there is always an $x \in K$ so that $\overline{\dim}_B K \leq \dim_A(K, x)$. However, in general one cannot hope for more than this: an example in [LR15] already has the property that $K \subset \mathbb{R}$ such that $\dim_A K = 1$ but $\dim_A(K, x) = 0$ for all $x \in K$ (see Example 2.10 for more detail); and moreover, in Example 2.11, we construct a compact set $K \subset \mathbb{R}$ with a point $x \in K$ so that $\dim_A(K, x) = 1$ but each $F \in \operatorname{Tan}(K, x)$ consists of at most two points.

However, many commonly studied families of "fractal" sets have a form of dynamical invariance, which is far from the case for general sets. As a result, it is of interest to determine general conditions under which the Assouad dimension is actually attained as the pointwise Assouad dimension at some point. To this end, we make the following definition.

Definition 1.1. We say that a compact set K is *self-embeddable* if for each $z \in K$ and $0 < r \leq \operatorname{diam} K$, there is a constant $a = a(z, r) > 0$ and a function $f\colon K \to B(z, r) \cap K$ so that

$$ar|x - y| \leq |f(x) - f(y)| \leq a^{-1} r|x - y|. \tag{1.1}$$

for all $x, y \in K$. We say that K is *uniformly self-embeddable* if the constant $a(z,r)$ can be chosen independently of z and r.

The class of self-embeddable sets is very broad and includes, for example, attractors of every possibly overlapping iterated function system $\{f_i\}_{i\in\mathcal{I}}$, where $\mathcal{I}$ is a finite index set and f_i is a strictly contracting bi-Lipschitz map from $\mathbb{R}^d$ to $\mathbb{R}^d$.

The class of *uniformly* self-embeddable sets includes the attractors of finite overlapping self-conformal iterated function systems. It is perhaps useful to compare uniform self-embeddability with quasi self-similarity, as introduced by Falconer [Fal89]. Our assumption is somewhat stronger since we also require the upper bound to hold in (1.1). This assumption is critical to our work since, in general, maps satisfying only the lower bound can decrease Assouad dimension. We also note that uniform self-embeddability is the primary assumption in [AKT20, Theorem 2.1].

Within this general class of sets, we establish the following result which guarantees the existence of at least one large tangent under self-embeddability, and an abundance of tangents under uniform self-embeddability.

Theorem A. *Let $K \subset \mathbb{R}^d$ be compact and self-embeddable. Then:*

(i) $\overline{\dim}_{\mathrm{B}} K \leq \dim_{\mathrm{A}}(K,x)$ *for all* $x \in K$.

(ii) There is an $x \in K$ and $F \in \mathrm{Tan}(K,x)$ so that $\mathcal{H}^{\dim_{\mathrm{A}} K}(F) > 0$. In particular, $\dim_{\mathrm{H}} F = \dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K$.

If K is uniformly self-embeddable, then there is a constant $c > 0$ so that

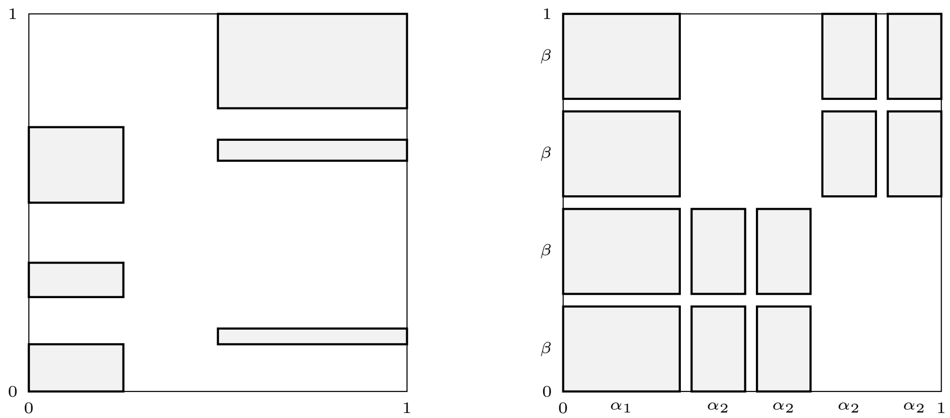
$$\dim_{\mathrm{H}}\{x \in K : \exists F \in \mathrm{Tan}(K,x) \text{ with } \mathcal{H}^{\dim_{\mathrm{A}} K}(F) \geq c\} = \dim_{\mathrm{H}} K. \tag{1.2}$$

The proof of Theorem A can be obtained by combining Theorem 2.12, Proposition 2.13, and Theorem 2.14. As a special case of the result for uniformly self-embeddable sets, suppose K is the attractor of a finite self-similar IFS in the real line with Hausdorff dimension $s < 1$. In this case there is a dichotomy: either $\mathcal{H}^s(K) > 0$, in which case K is Ahlfors–David regular, or $\dim_{\mathrm{A}} K = 1$. In particular, (1.2) cannot be improved in general to give a set with positive Hausdorff s-measure.

Beyond being of general interest, we believe this result will be a useful technical tool in the study of Assouad dimension for general attractors of bi-Lipschitz invariant sets. For instance, a common technique in studying attractors of iterated function systems is to relate the underlying geometry to symbolic properties associated with the coding space. Upper bounding the Hausdorff dimension of tangents is *a priori* easier since one may fix in advance a coding for the point. This is the situation, for example, in [BKR21, Theorem 5.2].

In Theorem A, we have established weak conditions which guarantee the existence of at least one large tangent, and relatively strong conditions which guarantee a set of points of full Hausdorff dimension with large tangents. A natural question to address is the following: to what extent do the results for uniformly self-embeddable sets extend to more general sets? Moreover, can we obtain even more precise information for specific families of sets?

With these questions in mind, we now turn our attention to two specific families of affine iterated function systems in the plane: specifically, the planar



(A) Gatzouras–Lalley (B) Barański

FIGURE 2. Generating maps associated with a Gatzouras–Lalley and Barański system. The parameters from the Barański carpet correspond to the example in Corollary 5.6 with $\delta = 1/40$.

self-affine carpets of *Gatzouras–Lalley* [LG92] and *Barański* [Bar07]. Note that these sets are self-embeddable but (except for some degenerate cases) not uniformly self-embeddable. We defer precise definitions and notation to §4.1; see Figure 2 for examples of the generating maps in these classes. In the following statement, let $\eta\colon \mathbb{R}^2 \to \mathbb{R}$ be the orthogonal projection onto the first coordinate axis and for $x \in \mathbb{R}^2$ let ℓ_x be the vertical line containing x.

Theorem B. *Let K be a Gatzouras–Lalley carpet. Then*

$$\mathcal{H}^{\dim_{\mathrm{H}} K}\big(\{x \in K : \dim_{\mathrm{A}}(K,x) \neq \dim_{\mathrm{A}} K\}\big) = 0.$$

On the other hand, for any $\dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{A}} K$,

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K,x) = \alpha\} = \dim_{\mathrm{H}} K.$$

Moreover, if $\eta(K)$ satisfies the SSC, then for any $x \in K$,
 (i) $\max\{\dim_{\mathrm{H}} F : F \in \operatorname{Tan}(K,x)\} = \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{A}} \ell_x \cap K$,
 (ii) $\dim_{\mathrm{A}}(K,x) = \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta(K) + \dim_{\mathrm{A}} \ell_x \cap K\}$.

Of course, if $\alpha \notin [\dim_{\mathrm{B}} K, \dim_{\mathrm{A}} K]$, then $\{x \in K : \dim_{\mathrm{A}}(K,x) = \alpha\} = \emptyset$. It follows immediately from Theorem B that

$$\dim_{\mathrm{A}}(K,x) = \max\{\dim_{\mathrm{H}} F : F \in \operatorname{Tan}(K,x)\}$$

if and only if $\dim_{\mathrm{A}} \ell_x \cap K \geq \dim_{\mathrm{B}} K - \dim_{\mathrm{B}} \eta(K)$. Moreover, if $s = \dim_{\mathrm{H}} K$, then $\mathcal{H}^s(K) > 0$ and furthermore $\mathcal{H}^s(K) < \infty$ if and only if K is Ahlfors–David regular (see [LG92]), in which case the results are trivial. We thus see that the majority of points, from the perspective of Hausdorff s-measure, have tangents with Hausdorff dimension attaining the Assouad dimension of K. However, we still have an abundance of points with pointwise Assouad dimension giving any other reasonable value.

The proof of Theorem B is obtained by combining Theorem 4.12 and Theorem 4.14. The dimensional results given in (i) and (ii) exhibit a precise version of a well-known phenomenon: at small scales, properly self-affine sets and measures look like products of the projection with slices. Note that, in order to obtain (i) and (ii), the strong separation condition in the projection is required or the pointwise Assouad dimension could be incorrect along sequences which are "arbitrarily close together at small scales". The formula holds for more general Gatzouras–Lalley carpets if one restricts attention to points where this does not happen (see Definition 4.3).

For Gatzouras–Lalley carpets with projection onto the first coordinate axis satisfying the strong separation condition, slices through x are precisely attractors of a non-autonomous iterated function system corresponding to the sequence of columns containing the point x (such a phenomenon was exploited in a more general setting in [FR22+]). In fact, as a pre-requisite to the proof of Theorem B, we establish a general formula for the Assouad dimension of non-autonomous self-similar sets satisfying the open set condition and with contraction ratios bounded uniformly from below. This is given in Theorem 3.7. The proof of Theorem 3.7, and indeed Theorem B, depend on some general properties of Assouad dimension which are elementary but may be of independent interest: certain subadditive regularity properties given in Corollary 3.5, and a reformulation of the Assouad dimension in terms of disc-packing bounds given in Proposition 3.6.

However, it turns out that the fact that Gatzouras–Lalley carpets have an abundance of large tangents does not extend to the non-dominated setting.

Theorem C. *There exists a Barański carpet K such that*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K.$$

In other words, the conclusion of Theorem A for uniformly self-embeddable sets does not necessarily extend beyond the uniformly self-embeddable case, even in the at first glance minor generalization consisting only of strictly diagonal self-affine functions acting in $\mathbb{R}^2$. The proof of Theorem C is given in Corollary 5.6, and it follows from a more general result describing when Barański carpets satisfying certain separation conditions have a large number of large tangents given in Theorem 5.4. This result can be found in Theorem 5.4, and it follows from more general formulas for the pointwise Assouad dimension at points which are coded by sequences which contract uniformly in one direction; see Proposition 5.3 for a precise formulation.

1.3. Some variants for future work.

Let $\phi\colon (0,1) \to (0,1)$ be a fixed function. We then define the *pointwise ϕ-Assouad dimension*, given by

$$\dim_{\mathrm{A}}^{\phi}(K,x) = \inf\Big\{s : \exists C > 0\, \forall 0 < r < 1 \\ N_{r^{1+\phi(r)}}\big(B(x,r) \cap K\big) \leq C r^{-\phi(r)s}\Big\}.$$

It is a straightforward to see that

$$\dim_{\mathrm{A}}^{\phi}(K,x)=\limsup_{r\to 0}\frac{\log N_{r^{1+\phi(r)}}\bigl(B(x,r)\cap K\bigr)}{\phi(r)\log(1/r)}.$$

The ϕ-Assouad dimensions are an example *dimension interpolation* [Fra21] and have been studied in detail in [BRT23+; GHM21]. In the specific case that $\phi(R)=\frac{1}{\theta}-1$ for some $\theta\in(0,1)$, this corresponds precisely to the Assouad spectrum [FY18] which (abusing notation) we may denote by $\dim_{\mathrm{A}}^{\theta}(K,x)$. In general, we expect the properties of the pointwise Assouad spectrum to be substantially different than the properties of the pointwise Assouad dimension.

For instance, in particular for Gatzouras–Lalley carpets, as $\theta\to 0$ one would expect to only witness the box dimension at every point in K, and as $\theta\to 1$ one would expect to witness the Hausdorff dimension of a maximal tangent, even when this quantity may be smaller than the pointwise Assouad dimension. In particular, it may happen that $\lim_{\theta\to 0}\dim_{\mathrm{A}}^{\theta}(K,x)>\lim_{\theta\to 1}\dim_{\mathrm{A}}^{\theta}(K,x)$. For intermediate values of θ, since the pointwise Assouad spectrum is determined by exponentially separated pairs of scales, it is likely that the value would depend in an essential way on the local dimensions of Bernoulli measures projected onto the first coordinate axis.

One might also consider the dual notion of the *pointwise lower dimension*, defined for $x\in K$ by

$$\begin{aligned}\dim_{\mathrm{L}}(K,x)=\sup\Bigl\{s:\exists C>0\,\exists\rho>0\,\forall 0<r\le R<\rho\\ N_r(B(x,R)\cap K)\ge C\Bigl(\frac{R}{r}\Bigr)^{s}\Bigr\}.\end{aligned}$$

It is established in [FHK+19] that the lower dimension may be analogously characterized as the minimum of Hausdorff dimensions of weak tangents. Therefore, a natural question is to ask if similar results hold for the pointwise lower dimension as well. However, the proofs we have given for Theorem A do not immediately translate to the case of the lower dimension since overlaps may increase dimension. On the other hand, the results concerning Gatzouras–Lalley carpets in Theorem B translate directly to the analogous lower dimension counterparts with appropriate modifications.

Finally, we note that an analogous notion for the pointwise Assouad dimension of measures was recently introduced in [Ant22+]. It would be interesting to investigate the relationship between these two notions of pointwise dimension.

1.4. Notation. Throughout, we work in $\mathbb{R}^d$ equipped with the usual Euclidean metric. Write $\mathbb{R}_+=(0,\infty)$. Given functions f and g, we say that $f\lesssim g$ if there is a constant $C>0$ so that $f(x)\le Cg(x)$ for all x in the domain of f and g. We write $f\approx g$ if $f\lesssim g$ and $g\lesssim f$.

2. TANGENTS AND POINTWISE ASSOUAD DIMENSION

2.1. Tangents and weak tangents. To begin this section, we precisely define the notions of tangent and weak tangent, and establish the fundamental relationship between the dimensions of tangents and the pointwise Assouad dimension.

Given a set $E \subset \mathbb{R}^d$ and $\delta > 0$, we denote the δ-neighbourhood of E by

$$E^{(\delta)} = \{x \in \mathbb{R}^d : \exists y \in E \text{ such that } |x - y| < \delta\}.$$

Now given a non-empty subset $X \subset \mathbb{R}^d$, we let $\mathcal{K}(X)$ denote the set of non-empty compact subsets of X equipped with the *Hausdorff metric*

$$d_{\mathcal{H}}(K_1, K_2) = \min\{p_{\mathcal{H}}(K_1; K_2), p_{\mathcal{H}}(K_2; K_1)\}$$

where

$$p_{\mathcal{H}}(K_1; K_2) = \inf\{\delta > 0 : K_1 \subset K_2^{(\delta)}\}.$$

If X is compact, then $(\mathcal{K}(X), d_{\mathcal{H}})$ is a compact metric space itself. We also write

$$\operatorname{dist}(E_1, E_2) = \sup\{\delta > 0 : E_1^{(\delta/2)} \cap E_2^{(\delta/2)} = \emptyset\}$$

for non-empty sets $E_1, E_2 \subset \mathbb{R}^d$.

We say that a set $F \in \mathcal{K}(B(0,1))$ is a *weak tangent* of $K \subset \mathbb{R}^d$ if there exists a sequence of similarity maps $(T_k)_{k=1}^{\infty}$ with $0 \in T_k(K)$ and similarity ratios λ_k diverging to infinity such that

$$F = \lim_{k\to\infty} T_k(K) \cap B(0,1)$$

in $\mathcal{K}(B(0,1))$. We denote the set of weak tangents of K by $\operatorname{Tan}(K)$. A key feature of the Assouad dimension is that it is characterized by Hausdorff dimensions of weak tangents. This result is originally from [KOR17, Proposition 5.7]. We refer the reader to [Fra20, Section 5.1] for more discussion on the context and history of this result.

Proposition 2.1 ([KOR17]). *We have*

$$\alpha := \dim_{\mathrm{A}} K = \max_{F \in \operatorname{Tan}(K)} \dim_{\mathrm{H}} F.$$

Moreover, the maximizing weak tangent F can be chosen so that $\mathcal{H}^{\alpha}(F) > 0$.

In a similar flavour, we say that F is a *tangent of K at $x \in K$* if there exists a sequence of similarity ratios $(\lambda_k)_{k=1}^{\infty}$ diverging to infinity such that

$$F = \lim_{k\to\infty} \lambda_k(K - x) \cap B(0,1)$$

in $\mathcal{K}(B(0,1))$. We denote the set of tangents of K at x by $\operatorname{Tan}(K, x)$.

Of course, $\mathrm{Tan}(K,x) \subset \mathrm{Tan}(K)$. Unlike in the case for weak tangents, we require the similarities in the construction of the tangent to in fact be homotheties. This choice is natural since, for example, a function $f\colon \mathbb{R} \to \mathbb{R}$ is differentiable at x if and only if the set of tangents of the graph of f at $(x, f(x))$ is the singleton $\{B(0,1) \cap \ell\}$ for some non-vertical line ℓ passing through the origin. In practice, compactness of the group of orthogonal transformations in $\mathbb{R}^d$ means this restriction will not cause any technical difficulties.

We observe that upper box dimensions of tangents provide a lower bound for the pointwise Assouad dimension.

Proposition 2.2. *For any compact set $K \subset \mathbb{R}^d$ and $x \in K$, $\dim_{\mathrm{A}}(K,x) \geq \overline{\dim}_{\mathrm{B}} F$ for any $F \in \mathrm{Tan}(K,x)$.*

Proof. Let $\alpha > \dim_{\mathrm{A}}(K,x)$ and suppose $F \in \mathrm{Tan}(K,x)$: we will show that $\overline{\dim}_{\mathrm{B}} F \leq \alpha$. First, get $C > 0$ such that for each $0 < r \leq R < 1$,

$$N_r(B(x,R) \cap K) \leq C\Big(\frac{R}{r}\Big)^{\alpha}.$$

Let $\delta > 0$ be arbitrary, and get a similarity T with similarity ratio λ such that $T(x) = 0$ and

$$d_{\mathcal{H}}(T(K) \cap B(0,1), F) \leq \delta.$$

Then there is a uniform constant $M > 0$ so that

$$M \cdot N_\delta(F) \leq N_\delta(T(K) \cap B(0,1)) = N_{\delta\lambda}(K \cap B(x,\lambda)) \leq C\Big(\frac{\lambda}{\delta\lambda}\Big)^{\alpha} = C\delta^{-\alpha}.$$

In other words, $\overline{\dim}_{\mathrm{B}} F \leq \alpha$. $\square$

One should not expect equality to hold in general: in Example 2.11, we construct an example of a compact set $K \subset \mathbb{R}$ and a point $x \in K$ so that $\dim_{\mathrm{A}}(K,x) = 1$ but every $F \in \mathrm{Tan}(K,x)$ consists of at most 2 points.

2.2. Level sets and measurability. We now make some observations concerning the multifractal properties of the function $x \mapsto \dim_{\mathrm{A}}(K,x)$. In particular, we are interested in the following quantities:

$$\mathcal{A}(K,\alpha) = \{x \in K : \dim_{\mathrm{A}}(K,x) = \alpha\} \qquad \text{and} \qquad \varphi(\alpha) = \dim_{\mathrm{H}} \mathcal{A}(K,\alpha).$$

We use the convention that $\dim_{\mathrm{H}} \emptyset = -\infty$. Observe that φ is a bi-Lipschitz invariant.

Let $\mathcal{K}(\mathbb{R}^d)$ denote the family of compact subsets of $\mathbb{R}^d$, equipped with the Hausdorff distance $d_{\mathcal{H}}$. We recall that $B(x,r)$ denotes the closed ball at x with radius r, and we let $B^\circ(x,r)$ denote the open ball at x with radius r. Given a compact set $K \subset \mathbb{R}^d$, we let $N_r^\circ(K)$ denote the minimal number of open sets with diameter r required to cover K, and $N_r^{\mathrm{pack}}(K)$ denote the size of a maximal centred packing of K by closed balls with radius r. Then, for $0 < r_1 \leq r_2$, we write

$$\mathcal{N}^\circ_{r_1,r_2}(K,x) = N^\circ_{r_1}(B(x,r_2)\cap K)$$
$$\mathcal{N}_{r_1,r_2}(K,x) = N^{\mathrm{pack}}_{r_1}(B^\circ(x,r_2)\cap K)$$

The following lemma is standard.

Lemma 2.3. *Fix $0 < r_1 \leq r_2$. Then:*
(i) $\mathcal{N}^\circ_{r_1,r_2}\colon \mathcal{K}(\mathbb{R}^d)\times\mathbb{R}^d \to [0,d]$ is lower semicontinuous.
(ii) $\mathcal{N}_{r_1,r_2}\colon \mathcal{K}(\mathbb{R}^d)\times\mathbb{R}^d \to [0,d]$ is upper semicontinuous.

We can use this lemma to establish the following fundamental measurability results.

Proposition 2.4. *The following measurability properties hold:*
(i) The function $(K,x)\mapsto \dim_{\mathrm{A}}(K,x)$ is Baire class 2.
(ii) $\mathcal{A}(K,\alpha)$ is Borel for any compact set K.

Proof. Since $\mathbb{R}^d$ is doubling,

$$\dim_{\mathrm{A}}(K,x) = \inf\Big\{s : \exists C>0\,\exists M\in\mathbb{N}\;\forall M\leq k\leq n \qquad \mathcal{N}_{2^{-n},2^{-k}}(K,x)\leq C2^{(n-k)s}\Big\}.$$

Equivalently, we may use $\mathcal{N}^\circ_{r_1,r_2}$ in place of $\mathcal{N}_{r_1,r_2}$. In particular,

$$\{(K,x) : \dim_{\mathrm{A}}(K,x) > s\} = \bigcap_{C=1}^\infty\bigcap_{M=1}^\infty\bigcup_{k=M}^\infty\bigcup_{n=k}^\infty(\mathcal{N}^\circ_{2^{-n},2^{-k}})^{-1}(C2^{(n-k)s},\infty).$$

and

$$\{(K,x) : \dim_{\mathrm{A}}(K,x) < t\} = \bigcup_{C\in\mathbb{Q}\cap(0,\infty)}\bigcup_{M=1}^\infty\bigcap_{k=M}^\infty\bigcap_{n=k}^\infty(\mathcal{N}_{2^{-n},2^{-k}})^{-1}(-\infty,C2^{(n-k)t}).$$

Thus $\{(K,x) : \dim_{\mathrm{A}}(K,x)\in(s,t)\}$ is a $G_{\delta\sigma}$-set, i.e. it is a countable union of sets expressible as a countable intersection of open sets, so $\dim_{\mathrm{A}}$ is Baire class 2.

Of course, the same argument also show that $x\mapsto\dim_{\mathrm{A}}(K,x)$ is Baire class 2 for a fixed compact set K, so that $\mathcal{A}(K,\alpha)$ is $G_{\delta\sigma}$ and, in particular, Borel. □

2.3. Tangents and pointwise dimensions of general sets. We now establish some general results on the existence of tangents for general sets. These results will also play an important technical role in the following sections: for many of our applications, it is not enough to have positive Hausdorff α-measure for $\alpha = \dim_{\mathrm{A}} K$, since in general Hausdorff α-measure does not interact well with the Hausdorff metric on $\mathcal{K}\big(B(0,1)\big)$.

Recall that the *Hausdorff α-content* of a set E is given by

$$\mathcal{H}^\alpha_\infty(E) = \inf\left\{\sum_{i=1}^\infty (\operatorname{diam} U_i)^\alpha : E \subset \bigcup_{i=1}^\infty U_i, U_i \text{ open}\right\}.$$

Of course, $\mathcal{H}^\alpha_\infty(E) \leq \mathcal{H}^\alpha(E)$ and $\mathcal{H}^\alpha_\infty(E) = 0$ if and only if $\mathcal{H}^\alpha(E) = 0$. We recall (see, e.g. [MM97, Theorem 2.1]) that $\mathcal{H}^\alpha_\infty$ is upper semicontinuous on $\mathcal{K}(B(0,1))$. Moreover, if $0 < \mathcal{H}^\alpha(E) < \infty$, then the density theorem for Hausdorff content implies that $\mathcal{H}^\alpha$-almost every $x \in E$ has a tangent with uniformly large Hausdorff α-content. We use these ideas in the following proofs.

We begin with a straightforward preliminary lemma which is proven, for example, in [KR16, Lemma 3.11].

Lemma 2.5. *Let $K \subset \mathbb{R}^d$ be compact. Then $\operatorname{Tan}(\operatorname{Tan}(K)) \subset \operatorname{Tan}(K)$.*

Proof. First suppose $E \in \operatorname{Tan}(K)$ and $F \in \operatorname{Tan}(E)$. Write $E = \lim_{n\to\infty} T_n(K) \cap B(0,1)$ and $F = \lim_{n\to\infty} S_n(E) \cap B(0,1)$ for some sequences of similarities (T_n) and (S_n) with similarity ratios diverging to infinity. For each $\epsilon > 0$, let N be sufficiently large so that

$$d_{\mathcal{H}}(S_N(E) \cap B(0,1), F) \leq \frac{\epsilon}{2}.$$

Suppose S_N has similarity ratio λ_N, and let M be sufficiently large so that

$$d_{\mathcal{H}}(T_M(K) \cap B(0,1), E) \leq \frac{\epsilon}{2\lambda_N}$$

It follows that

$$d_{\mathcal{H}}(S_N \circ T_M(K) \cap B(0,1), F) \leq \epsilon.$$

But $\epsilon > 0$ was arbitrary, as required. □

Now, given a set with positive and finite Hausdorff measure, we can always find a tangent with large Hausdorff content.

Lemma 2.6. *Let $K \subseteq \mathbb{R}^d$ be a compact set with $0 < \mathcal{H}^\alpha(K) < \infty$. Then for $\mathcal{H}^\alpha$-almost every $x \in K$, there is an $F \in \operatorname{Tan}(K, x)$ such that $\mathcal{H}^\alpha_\infty(F) \geq 1$.*

Proof. By the same proof as [Mat95, Theorem 6.2], for $\mathcal{H}^\alpha$-almost every $x \in K$, there is a sequence of scales $(r_n)_{n=1}^\infty$ converging to zero such that

$$1 \leq \lim_{n\to\infty} r_n^{-\alpha}\mathcal{H}^\alpha_\infty\big(B(x,r_n) \cap K\big).$$

Then

$$\mathcal{H}^\alpha_\infty\big(r_n^{-1}(K - x) \cap B(0,1)\big) = r_n^{-\alpha}\mathcal{H}^\alpha_\infty\big(B(x,r_n) \cap K\big) \xrightarrow{n\to\infty} 1.$$

But Hausdorff α-content is upper semicontinuous, so passing to a subsequence if necessary,

$$F = \lim_{n\to\infty} \left(r_n^{-1}(K-x)\cap B(0,1)\right)$$

satisfies $\mathcal{H}^\alpha_\infty(F) \geq 1$. □

Of course, we can combine the previous two results to obtain the following improvement of Proposition 2.1.

Corollary 2.7. *Let K be a compact set with $\dim_{\mathrm{A}} K = \alpha$. Then there is a weak tangent $F \in \operatorname{Tan}(K)$ with $\mathcal{H}^\alpha_\infty(F) \geq 1$.*

Proof. By Proposition 2.1, there is $E \in \operatorname{Tan}(K)$ such that $\mathcal{H}^\alpha(E) > 0$. By [Fal90, Theorem 4.10], there is a compact $E' \subset E$ such that $0 < \mathcal{H}^\alpha(E') < \infty$. Then by Lemma 2.6, there is $F' \in \operatorname{Tan}(E')$ with $\mathcal{H}^\alpha_\infty(F') \geq 1$. But $F' \subset F$ for some $F \in \operatorname{Tan}(E)$, and by Lemma 2.5, $F \in \operatorname{Tan}(K)$ with $\mathcal{H}^\alpha_\infty(F) \geq \mathcal{H}^\alpha_\infty(F') \geq 1$. □

We now establish bounds on the pointwise Assouad dimension and tangents for general sets.

Proposition 2.8. *Let $K \subset \mathbb{R}^d$. Then:*

(i) If K is analytic, for any s such that $\mathcal{H}^s(K) > 0$, there is a set $E \subset K$ with $\mathcal{H}^s(E) > 0$ so that for each $x \in E$, there is a tangent $F \in \operatorname{Tan}(\overline{K}, x)$ with $\mathcal{H}^s_\infty(F) \geq 1$.

(ii) If K is compact, there is an $x \in K$ such that $\dim_{\mathrm{A}}(K, x) \geq \overline{\dim}_{\mathrm{B}} K$.

Proof. The proof of (i) follows directly from Lemma 2.6, recalling that we can always find a compact subset $E \subset K$ such that $0 < \mathcal{H}^s(E) < \infty$ (combine [Mat95, Theorem 8.19] and [BP17, Corollary B.2.4]).

We now see (ii). Let $\overline{\dim}_{\mathrm{B}} K = t$. We first observe that for any $r > 0$, there is an $x \in K$ so that $\overline{\dim}_{\mathrm{B}} B(x,r) \cap K = t$. In particular, we may inductively construct a nested sequence of balls $B(x_k, r_k)$ with $\lim_{k\to\infty} r_k = 0$ so that $\overline{\dim}_{\mathrm{B}} K \cap B(x_k, r_k) = t$ for all $k \in \mathbb{N}$. Since K is compact, take $x = \lim_{k\to\infty} x_k \in K$. We verify that $\dim_{\mathrm{A}}(K, x) \geq t$. Let $C > 0$ and $\rho > 0$ be arbitrary. Since the x_k converge to x and the r_k converge to 0, get some k so that $B(x_k, r_k) \subset B(x, \rho)$. Thus for any $\epsilon > 0$ and $r > 0$ sufficiently small depending on ϵ,

$$N_r\left(B(x,\rho)\cap K\right) \geq N_r\left(B(x_k, r_k)\cap K\right) \geq C\left(\frac{r_k}{r}\right)^{t-\epsilon}.$$

Thus $\dim_{\mathrm{A}}(K, x) \geq t$. □

Remark 2.9. Note that compactness is essential in Proposition 2.8 (ii) since there are sets with $\overline{\dim}_{\mathrm{B}} K = 1$ but every point is isolated: consider, for instance, the set $E = \{(\log n)^{-1} : n = 2, 3, \ldots\}$. In this case, $\overline{E} = E \cup \{0\}$ and $\dim_{\mathrm{A}}(\overline{E}, 0) = 1$. This example also shows that (ii) can hold with exactly 1 point.

Finally, we construct some general examples which go some way to showing that the results for general sets given in this section are sharp.

Example 2.10. In general, the Assouad dimension can only be characterized by weak tangents rather than by tangents. For example, consider the set K from [LR15, Example 2.20], defined by

$$K = \{0\} \cup \left\{2^{-k} + \ell 4^{-k} : k \in \mathbb{N}, \ell \in \{0, 1, \dots, k\}\right\}$$

Since K contains arithmetic progressions of length k for all $k \in \mathbb{N}$, $\dim_{\mathrm{A}} K = 1$. However, $\dim_{\mathrm{A}}(K, x) = 0$ for all $x \in K$ and, therefore, by Proposition 2.2, $\dim_{\mathrm{H}} F = 0$ for all $F \in \operatorname{Tan}(K, x)$ and $x \in K$.

Example 2.11. We give an example of a compact set K and a point $x \in K$ so that $\dim_{\mathrm{A}}(K, x) = 1$ but each $F \in \operatorname{Tan}(K, x)$ consists of at most finitely many points.

Set $a_k = 4^{-k^2}$ and observe that $k a_{k+1}/a_k \leq 1/k$. For each $k \in \mathbb{N}$, write $\ell_k = \lfloor 2^k/k \rfloor$ and set

$$K = \{0\} \cup \bigcup_{k=1}^{\infty} \left\{ a_k \frac{2^k - \ell_k}{2^k}, a_k \frac{2^k - \ell_k - 1}{2^k}, \dots, a_k \right\}$$

and consider the point $x = 0$. First observe for all $\epsilon > 0$ and all k sufficiently small depending on ϵ,

$$N_{2^{-k} \cdot a_k}\big(B(0, a_k) \cap K\big) \geq \frac{\ell_k}{2} \geq 2^{(1-\epsilon)k}$$

which gives that $\dim_{\mathrm{A}}(K, 0) = 1$.

On the other hand, for $k \in \mathbb{N}$,

$$a_k^{-1} K \cap B(0, 1) \subset [0, a_{k+1}/a_k] \cup [1/k, 1].$$

Since $k a_{k+1}/a_k \leq 1/k$, it follows that for any $\lambda \geq 1$ and $\lambda K \cap B(0, 1)$ can be contained in a union of two intervals with arbitrarily small length as λ diverges to ∞. Thus any tangent $F \in \operatorname{Tan}(K, 0)$ consists of at most 2 points.

2.4. Tangents of dynamically invariant sets. We recall from Proposition 2.8 (ii) that the Assouad dimension of K need not be attained as the Assouad dimension of a point, and even the Assouad dimension at a point need not be attained as the upper box dimension of a tangent at that point.

Now recall the definition of self-embeddability from Definition 1.1. For self-embeddable sets, we can prove directly that the Assouad dimension of K is attained as the Hausdorff dimension of a tangent. In fact, the tangent can be chosen to have positive $\mathcal{H}^{\alpha}$-measure for $\alpha = \dim_{\mathrm{A}} K$.

Theorem 2.12. *Let $K \subseteq \mathbb{R}^d$ be compact and self-embeddable with $\alpha = \dim_{\mathrm{A}} K$. Then there is a dense set of points $x \in K$ for which there exist $F \in \operatorname{Tan}(K, x)$ such that $\mathcal{H}^{\alpha}_{\infty}(F) \gtrsim_{\alpha} 1$.*

Proof. By self-embeddability and since $\dim_{\mathrm{A}}(K, x) = \dim_{\mathrm{A}}(f(K), f(x))$ for a bi-Lipschitz map f, it suffices to construct a single point x with this property.

First, begin with some arbitrary ball $B(x_1, r_1)$ with $x_1 \in K$ and $0 < r_1 \leq 1$. Since K is self-embeddable, get a bi-Lipschitz map $f_1 \colon K \to K \cap B(x_1, r_1)$. Since $\dim_{\mathrm{A}} f_1(K) = \alpha$, by Corollary 2.7 there is a weak tangent F_1 of $f_1(K)$ such that $\mathcal{H}^\alpha_\infty(F_1) \geq 1$. Since F_1 is a weak tangent of $f_1(K)$, there is a similarity T_1 with similarity ratio $\lambda_1 \geq 1$ such that $0 \in T_1(K)$ and

$$d_{\mathcal{H}}\big(T_1(f_1(K)) \cap B(0,1), F_1\big) \leq 1.$$

Then choose $x_2 \in K$ and $r_2 \leq 1/2$ so that $B(x_2, r_2) \subset T_1^{-1} B^\circ(0,1)$.

Repeating the above construction, next with the ball $B(x_2, r_2)$, and iterating, we obtain a sequence of similarity maps $(T_n)_{n=1}^\infty$ each with similarity ratio $\lambda_n \geq n$, bi-Lipschitz maps f_n, and compact sets F_n such that

1. $T_{n+1}^{-1} B(0,1) \subseteq T_n^{-1} B(0,1)$,
2. $d_{\mathcal{H}}\big(T_n(f_n(K)) \cap B(0,1), F_n\big) \leq \dfrac{1}{n}$, and
3. $\mathcal{H}^\alpha_\infty(F_n) \geq 1$.

Let $x = \lim_{n\to\infty} T_n^{-1}(0)$ and note by 1 that $x \in T_n^{-1} B(0,1)$ for all $n \in \mathbb{N}$. Let h_n be a similarity with similarity ratio $1/2$ such that

$$d_{\mathcal{H}}\Big(\frac{\lambda_n}{2}(f_n(K) - x) \cap B(0,1), h_n(F_n)\Big) \leq \frac{1}{n}.$$

Observe that $\mathcal{H}^\alpha_\infty(h_n(F_n)) \geq 2^{-\alpha}$. Thus passing to a subsequence if necessary, since $f_n(K) \subseteq K$, we may set

$$F_0 = \lim_{n\to\infty} \frac{\lambda_n}{2}(f_n(K) - x) \cap B(0,1) \qquad \text{and} \qquad F = \lim_{n\to\infty} \frac{\lambda_n}{2}(K - x) \cap B(0,1).$$

and observe that $F_0 \subseteq F$. Again passing to a subsequence if necessary, by compactness of the orthogonal group, 2 and the triangle inequality, there is an isometry h so that $\lim_{n\to\infty} h \circ h_n(F_n) = F_0$. Thus by upper semicontinuity of Hausdorff content,

$$\mathcal{H}^\alpha_\infty(F) \geq \mathcal{H}^\alpha_\infty(F_0) \geq \lim_{n\to\infty} \mathcal{H}^\alpha_\infty(h \circ h_n(F_n)) = 2^{-\alpha}$$

as required. □

We recall from Proposition 2.8 (ii) that, for a general compact set K, the upper box dimension of K provides a lower bound for the pointwise Assouad dimension at *some* point. For self-embeddable sets, we observe that the upper box dimension provides a uniform lower bound for the pointwise Assouad dimension at *every* point. On the other hand, the upper box dimension *does not* lower bound the maximal dimension of a tangent. For an example of this phenomenon, see Theorem 4.12.

Proposition 2.13. *Let $K \subseteq \mathbb{R}^d$ be self-embeddable. Then for any $x \in K$, we have* $\dim_{\mathrm{A}}(K, x) \geq \overline{\dim}_{\mathrm{B}} K$.

Proof. Fix $\alpha < \overline{\dim}_{\mathrm{B}} K$ and $x \in K$. Let $C > 0$ and $\rho > 0$ be arbitrary. Since K is self-embeddable, there is some bi-Lipschitz map $f\colon K \to B(x,\rho)$ so that $f(K) \subseteq K$. Since $\overline{\dim}_{\mathrm{B}} f(K) > \alpha$, there is some $0 < r \leq \rho$ so that

$$N_r(B(x,\rho) \cap K) \geq N_r(f(K)) \geq C\Big(\frac{\rho}{r}\Big)^{\alpha}.$$

Since $C > 0$ and $\rho > 0$ were arbitrary, $\dim_{\mathrm{A}}(K,x) \geq \alpha$, as required. □

Now assuming uniform self-embeddability, we will see that the set of points with tangents that have positive $\mathcal{H}^\alpha$-measure has full Hausdorff dimension for $\alpha = \dim_{\mathrm{A}} K$. Since uniformly self-embeddable sets satisfy the hypotheses of [Fal89, Theorem 4], it always holds that $\overline{\dim}_{\mathrm{B}} K = \dim_{\mathrm{H}} K$ (see also [Fra14, Theorem 2.10]). On the other hand, it can happen in this class of sets that $\overline{\dim}_{\mathrm{B}} K < \alpha$: for example, this is the situation for self-similar sets in $\mathbb{R}$ with $\overline{\dim}_{\mathrm{B}} K < 1$ which fail the weak separation condition; see [FHO+15, Theorem 1.3]. We provide a subset of full Hausdorff dimension for which each point has a tangent with positive Hausdorff α-measure.

The idea of the proof is essentially as follows. Let F be a weak tangent for K with strictly positive Hausdorff α-content. For each $s < \overline{\dim}_{\mathrm{B}} K$, using the implicit method of [Fal89, Theorem 4], we can construct a well-distributed set of N balls at resolution δ, where $\delta^{-s} \ll N$. Then, inside each ball, using uniform self-embeddability, we can map an image of an approximate tangent $T_\delta^{-1}(B(0,1)) \cap K \approx F$ where T_δ has similarity ratio λ. Choosing N to be large, the resulting collection of images of the approximate tangent F is again a family of well-distributed balls at resolution $\lambda^{-1}\delta$, with $(\lambda^{-1}\delta)^{-s} \approx N$. Repeating this construction along a sequence of tangents converging to F yields a set E with $\dim_{\mathrm{H}} E \geq s$ such that each $x \in E$ has a tangent which is an image of F (up to some negligible distortion), which has positive Hausdorff α-content by upper semicontinuity of content.

We fix a compact set K. To simplify notation, we say that a function $f\colon K \to K$ is in $\mathcal{G}(z,r,c)$ for $z \in K$ and $c, r > 0$ if $f(K) \subset B(z,r)$ and

$$cr|x-y| \leq |f(x)-f(y)| \leq c^{-1}r|x-y|$$

for all $x, y \in K$.

Theorem 2.14. *Let $K \subset \mathbb{R}^d$ be uniformly self-embeddable and let $\alpha = \dim_{\mathrm{A}} K$. Then*

$$\dim_{\mathrm{H}}\{x \in K : \exists F \in \operatorname{Tan}(K,x) \text{ with } \mathcal{H}^\alpha_\infty(F) \gtrsim 1\} = \dim_{\mathrm{H}} K = \overline{\dim}_{\mathrm{B}} K.$$

Proof. Write $\alpha = \dim_{\mathrm{A}} K$. If $\overline{\dim}_{\mathrm{B}} K = 0$ we are done; otherwise, let $0 < s < \overline{\dim}_{\mathrm{B}} K$ be arbitrary. Since K is uniformly self-embeddable, there is a constant $a \in (0,1)$ so that for each $z \in K$ and $0 < r \leq \operatorname{diam} K$ there is a map $f \in \mathcal{G}(z,r,a)$. Next, from Corollary 2.7, there is a compact set $F \subset B(0,1)$ with $\mathcal{H}^\alpha_\infty(F) \geq 1$ and a sequence of similarities $(T_k)_{k=1}^\infty$ with similarity ratios $(\lambda_k)_{k=1}^\infty$ such that

$$F = \lim_{k\to\infty} T_k(K) \cap B(0,1)$$

with respect to the Hausdorff metric. Set $Q_k = T_k^{-1}(B(0,1)) \cap K$. We will construct a Cantor set $E \subset K$ of points each of which has pointwise Assouad dimension at least α and has $\dim_{\mathrm{H}} E \geq s$.

We begin with a preliminary construction. First, since $s < \overline{\dim}_{\mathrm{B}} K$, there is some $r_0 > 0$ and a collection of points $\{y_i\}_{i=1}^{N_0} \subset K$ such that $|y_i - y_j| > 3r_0$ for all $i \neq j$ and $N_0 \geq 2^s a^{-s} r_0^{-s}$. Now for each i, take a map $\phi_i \in \mathcal{G}(y_i, r_0, a)$. Write $\mathcal{I} = \{1, \ldots, N_0\}$, and for $\mathtt{i} = (i_1, \ldots, i_n) \in \mathcal{I}^n$ set

$$\phi_{\mathtt{i}} = \phi_{i_1} \circ \cdots \circ \phi_{i_n},$$

and, having fixed some $x_0 \in K$, write $x_{\mathtt{i}} = \phi_{\mathtt{i}}(x_0) \in \phi_{\mathtt{i}}(K)$. Observe that if the maximal length of a common prefix of $\mathtt{i}_1$ and $\mathtt{i}_2$ is m, then

$$\operatorname{dist}(\phi_{\mathtt{i}}(K), \phi_{\mathtt{j}}(K)) \geq r_0(ar_0)^m.$$

We now begin our inductive construction. Without loss of generality, we may assume that $\lambda_n \geq 12$ for all $n \in \mathbb{N}$ and $r_0 \leq 1$. First, for each $n \in \mathbb{N}$, define constants $(m_n)_{n=1}^\infty \subset \{0\} \cup \mathbb{N}$ and $(\rho_n)_{n=1}^\infty$ converging monotonically to zero from above by the rules

1. $2^{-m_n} \leq \dfrac{a^2 r_0 \lambda_n^{-1}}{3}$,
2. $\rho_0 = \operatorname{diam} K$, and
3. $\rho_n \coloneqq \rho_{n-1} \cdot \dfrac{a\lambda_n^{-1} \cdot (ar_0)^{m_n}}{3}$.

Next, for $n \in \mathbb{N} \cup \{0\}$ we inductively choose points $y_{n,\mathtt{i}} \in K$ and maps $\Psi_{n,\mathtt{i}} \in \mathcal{G}(y_{n,\mathtt{i}}, \rho_n, a)$ for $\mathtt{i} \in \mathcal{I}^{m_1} \times \cdots \times \mathcal{I}^{m_n}$. Let $\varnothing$ denote the empty word and let $y_{0,\varnothing} \in K$ be arbitrary and let $\Psi_{0,\varnothing}$ denote the identity map. Then for each $\mathtt{k} = \mathtt{ij}$ with $\mathtt{i} \in \mathcal{I}^{m_1} \times \cdots \times \mathcal{I}^{m_{n-1}}$ and $\mathtt{j} \in \mathcal{I}^{m_n}$, sequentially choose:

4. $\psi_{n,\mathtt{k}} \in \mathcal{G}(\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}}), \rho_n \lambda_n a^{-1}, a)$
5. $y_{n,\mathtt{k}} = \psi_{n,\mathtt{k}} \circ T_n^{-1}(0)$
6. $\Psi_{n,\mathtt{k}} \in \mathcal{G}(y_{n,\mathtt{k}}, \rho_n, a)$

Finally, write $\mathcal{J}_0 = \{\varnothing\}$, $\mathcal{J}_n = \mathcal{I}^{m_1} \times \cdots \times \mathcal{I}^{m_n}$ for $n \in \mathbb{N}$, and let

$$E_n = \bigcup_{\mathtt{i} \in \mathcal{J}_n} B(y_{n,\mathtt{k}}, 3\rho_n) \qquad \text{and} \qquad E = K \cap \bigcap_{n=1}^\infty E_n.$$

Suppose $\mathtt{i} \in \mathcal{J}_{n-1}$ and $\mathtt{j} \in \mathcal{I}^{m_n}$. Since $x_{\mathtt{j}} \in K$, $\Psi_{n-1,\mathtt{i}}(K) \subset B(y_{n-1,\mathtt{i}}, \rho_{n-1})$, and $y_{n,\mathtt{ij}} \in \psi_{n,\mathtt{ij}}(K) \subset B(\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}}), \rho_{n-1})$, we conclude since $\rho_{n-1} \geq 3\rho_n$ that

$$B(y_{n,\mathtt{ij}}, 3\rho_n) \subset B(y_{n-1,\mathtt{i}}, 3\rho_{n-1}).$$

Moreover, $y_{n,\mathtt{ij}} \in K$, so the sets E_n are non-empty nested compact sets and therefore E is non-empty.

We next observe the following fundamental separation properties of the balls in the construction of the sets E_n. Let $n \in \mathbb{N}$ and suppose $\mathtt{j}_1 \neq \mathtt{j}_2$ in $\mathcal{I}^{m_n}$ and $\mathtt{i} \in \mathcal{J}_{n-1}$ (writing $\mathcal{J}_0 = \{\varnothing\}$). Suppose $\mathtt{j}_1$ and $\mathtt{j}_2$ have a common prefix of maximal length m. First recall that $|x_{\mathtt{j}_1} - x_{\mathtt{j}_2}| \geq r_0(ar_0)^m$, so that

$$|\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}_1}) - \Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}_2})| \geq \rho_{n-1}(ar_0)^{m+1}.$$

Then, since for $j = 1, 2$

$$y_{n,\mathtt{i}\mathtt{j}_j} \in \psi_{n,\mathtt{i}\mathtt{j}_j}(K) \subset B\Big(\Psi_{n-1,\mathtt{i}}(x_{\mathtt{j}_j}), \frac{\rho_{n-1}(ar_0)^{m_n}}{3}\Big)$$

we observe that

$$|y_{n,\mathtt{i}\mathtt{j}_1} - y_{n,\mathtt{i}\mathtt{j}_2}| \geq \rho_{n-1}(ar_0)^{m+1} - 2\frac{\rho_{n-1}(ar_0)^{m_n}}{3} \geq \frac{\rho_{n-1}(ar_0)^{m+1}}{3}.$$

Since we assumed that $\lambda_n \geq 12$, by the triangle inequality

$$\operatorname{dist}\big(B(y_{n,\mathtt{i}\mathtt{j}_1}, 3\rho_n), B(y_{n,\mathtt{i}\mathtt{j}_2}, 3\rho_n)\big) \geq \frac{\rho_{n-1}(ar_0)^{m+1}}{3} - 6\rho_n \geq \frac{\rho_{n-1}(ar_0)^{m+1}}{6}. \tag{2.1}$$

We first show that $\dim_{\mathrm{H}} E \geq s$. By the method of repeated subdivision, define a Borel probability measure μ with $\operatorname{supp}\mu = E$ and for $\mathtt{i} \in \mathcal{J}_n$,

$$\mu(B(y_{n,\mathtt{i}}, 3\rho_n) \cap K) = \frac{1}{\#\mathcal{J}_n}.$$

Now suppose U is an arbitrary open set with $U \cap E \neq \emptyset$. Intending to use the mass distribution principle, we estimate $\mu(U)$. Assuming that U has sufficiently small diameter, let $n \in \mathbb{N}$ be maximal so that

$$\operatorname{diam} U \leq \frac{a^{-1}\lambda_n}{2}\rho_n = \frac{\rho_{n-1}(ar_0)^{m_n}}{6}.$$

By (2.1), there is a unique $\mathtt{i} \in \mathcal{J}_n$ such that $U \cap B(y_{n,\mathtt{i}}, 3\rho_n) \neq \varnothing$. We first recall by choice of the constants m_n that

$$\begin{aligned}\rho_n &= (\operatorname{diam} K) \cdot \Big(\frac{a^2 r_0 \lambda_n^{-1}}{3}\Big)^n (ar_0)^{m_1+\cdots+m_n} \\ &\geq (\operatorname{diam} K) 2^{-(m_1+\cdots+m_n)}(ar_0)^{m_1+\cdots+m_n}.\end{aligned}$$

There are two cases. First assume $\rho_n/6 < \operatorname{diam} U$. Thus

$$\begin{aligned}\mu(U) &\leq \frac{1}{\#\mathcal{J}_n} \leq \Big(\frac{1}{2}ar_0\Big)^{s(m_1+\cdots+m_n)} \leq (\operatorname{diam} K)^{-s}\rho_n^s \\ &\leq \Big(\frac{6}{\operatorname{diam} K}\Big)^s \cdot (\operatorname{diam} U)^s.\end{aligned}$$

Otherwise, let $k \in \{0, \ldots, m_{n+1} - 1\}$ be so that

$$\frac{\rho_n(ar_0)^{k+1}}{6} < \operatorname{diam} U \leq \frac{\rho_n(ar_0)^k}{6}.$$

By (2.1), U intersects at most $N_0^{m_n - k}$ balls $B(y_{n+1,\omega}, 3\rho_{n+1})$ for $\omega \in \mathcal{J}_{n+1}$, so since $2^{-sk} \leq 1$,

$$\mu(U) \leq \frac{1}{\#\mathcal{J}_n \cdot N_0^k} \leq (\operatorname{diam} K)^{-s}\rho_n^s \cdot (2^{-s}(ar_0)^s)^k$$

$$\leq \Big(\frac{6}{ar_0 \operatorname{diam} K}\Big)^s \cdot \Big(\frac{\rho_n (ar_0)^{k+1}}{6}\Big)^s$$
$$\leq \Big(\frac{6}{ar_0 \operatorname{diam} K}\Big)^s \cdot \big(\operatorname{diam} U\big)^s.$$

This treats all possible small values of $\operatorname{diam} U$, so there is a constant $M > 0$ such that $\mu(U) \leq M(\operatorname{diam} U)^s$. Thus $\dim_{\mathrm{H}} E \geq s$ by the mass distribution principle.

Now fix

$$C = (3 + a^{-2})^{-\alpha}.$$

We will show that each $z \in E$ has a tangent with Hausdorff α-content at least C. Let $z \in E$ and define

$$S_n(x) = \frac{x - z}{\rho_n(3 + a^{-2})}.$$

Our tangent will be an accumulation point of the sequence $(S_n(K) \cap B(0,1))_{n=1}^{\infty}$. Now fix $n \in \mathbb{N}$. Since $z \in E$, there is some $\omega \in \mathcal{J}_n$ so that $z \in B(y_{n,\omega}, 3\rho_n)$. By choice of $y_{n,\omega}$, $Q_n = B\big(\psi_{n,\omega}^{-1}(y_{n,\omega}), \lambda_n^{-1}\big) \cap K$ so that

$$\psi_{n,\omega}(Q_n) \subseteq B\big(y_{n,\omega}, \rho_n a^{-2}\big) \cap K \subseteq B\big(z, \rho_n(3 + a^{-2})\big) \cap K$$

and therefore, writing $\Phi_n = S_n \circ \psi_{n,\omega} \circ T_n^{-1}$,

$$\Phi_n(T_n(K) \cap B(0,1)) \subset S_n(K) \cap B(0,1).$$

Then for $x, y \in T_n(K) \cap B(0,1)$, by the choice of ψ in (4),

$$\frac{|x - y|}{3 + a^{-2}} \leq |\Phi_n(x) - \Phi_n(y)| \leq \frac{|x - y|}{a^2(3 + a^{-2})}. \tag{2.2}$$

Now, passing to a subsequence $(n_k)_{k=1}^{\infty}$, we can ensure that

$$\lim_{k\to\infty} \Phi_{n_k}(F) = Z_0 \qquad \text{and} \qquad \lim_{k\to\infty} S_{n_k}(K) \cap B(0,1) = Z.$$

Moreover, recall that $\lim_{k\to\infty} T_{n_k}(K) \cap B(0,1) = F$ and $\mathcal{H}^{\alpha}_{\infty}(F) \geq 1$. Observe by (2.2) that $\mathcal{H}^{\alpha}_{\infty}(\Phi_{n_k}(F)) \geq C$ for each k, so by upper semicontinuity of Hausdorff content, $\mathcal{H}^{\alpha}_{\infty}(Z_0) \geq C$. But again by (2.2),

$$d_{\mathcal{H}}\big(Z_0, \Phi_{n_k}(T_{n_k}(K) \cap B(0,1))\big) \leq d_{\mathcal{H}}(Z_0, \Phi_{n_k}(F)) + \frac{d_{\mathcal{H}}\big(F, T_{n_k}(K) \cap B(0,1)\big)}{a^2(3 + a^{-2})}$$

so in fact $Z_0 \subset Z$ and $\mathcal{H}^{\alpha}_{\infty}(Z) \geq C$, as claimed. □

Remark 2.15. We note that the upper distortion bound in the definition of uniform self-embeddability is used only at the very last step to guarantee that the images $\Phi_{n_k}(T_{n_k}(K) \cap B(0,1))$ converge to a large set whenever the $T_{n_k}(K) \cap B(0,1)$ converge to a large set.

3. ASSOUAD DIMENSION OF NON-AUTONOMOUS SELF-SIMILAR SETS

3.1. Non-autonomous self-similar sets. The notion of a non-autonomous self-conformal set was introduced and studied in [RU16], where under certain regularity assumptions the authors prove that the Hausdorff and box dimensions are equal and given by the zero of a certain pressure function. In this section, we consider a special case of their construction. For each $n \in \mathbb{N}$, let $\Phi_n = \{S_{n,j}\}_{j \in \mathcal{J}_n}$ be a finite family of similarity maps $S_{n,j} \colon \mathbb{R}^d \to \mathbb{R}^d$ of the form

$$S_{n,j}(\boldsymbol{x}) = r_{n,j} O_{n,j} \boldsymbol{x} + \boldsymbol{d}_{n,j}$$

where $r_{n,j} \in (0,1)$ and $O_{n,j}$ is an orthogonal matrix. To avoid degenerate situations, we assume that associated with the sequence $(\Phi_n)_{n=1}^{\infty}$ there is an invariant compact set $X \subset \mathbb{R}^d$ (that is $S_{n,j}(X) \subset X$ for all $n \in \mathbb{N}$ and $j \in \mathcal{J}_n$) and moreover that

$$\lim_{n \to \infty} \sup\{r_{1,j_1} \cdots r_{n,j_n} : j_i \in \mathcal{J}_i \text{ for each } i = 1, \dots, n\} = 0. \tag{3.1}$$

Under these assumptions, associated with the sequence $(\Phi_n)_{n=1}^{\infty}$ is an *attractor*

$$K = \bigcap_{n=1}^{\infty} \bigcup_{(j_1,\dots,j_n) \in \mathcal{J}_1 \times \dots \times \mathcal{J}_n} S_{1,j_1} \circ \dots \circ S_{n,j_n}(X).$$

Since X is compact and invariant under any map $S_{n,j}$ with $j \in \mathcal{J}_n$, finiteness of each $\mathcal{J}_n$ implies that K is the intersection of a nested sequence of compact sets and therefore non-empty and compact. The sequence $(\Phi_n)_{n=1}^{\infty}$ is called a *non-autonomous iterated function system (IFS)* and the attractor K is called the *non-autonomous self-similar set*. We refer the reader to [RU16, §2] for more detail on this construction in a general setting.

Definition 3.1. We say that the non-autonomous IFS $(\Phi_n)_{n=1}^{\infty}$

(i) satisfies the *open set condition* if the invariant compact set X can be chosen to have non-empty interior $U = X^\circ$ so that for each $n \in \mathbb{N}$ and $j, j' \in \mathcal{J}_n$, $S_{n,j}(U) \subset U$ and $S_{n,j}(U) \cap S_{n,j'}(U) = \emptyset$ for $j \neq j' \in \mathcal{J}_n$; and

(ii) has *uniformly bounded contractions* if there is an $r_{\min} > 0$ so that $r_{\min} \leq r_{n,j}$ for all $n \in \mathbb{N}$ and $j \in \mathcal{J}_n$.

Since $\mathrm{Leb}\big(\sum_{j \in \mathcal{J}_n} S_{n,j}(U)\big) \leq \mathrm{Leb}(U)$ and $\mathrm{Leb}(S_{n,j}(U)) \geq r_{\min}^d > 0$, the above two conditions combine to give the following additional condition:

(iii) There is an $M \in \mathbb{N}$ so that $\#\mathcal{J}_n \leq M$ for all $n \in \mathbb{N}$.

Our main goal in this section is, assuming the open set condition and uniformly bounded contractions, to establish an explicit formula for $\dim_{\mathrm{A}} K$, depending only on the $r_{n,j}$. This will be done in Theorem 3.7. In order to obtain this result, we first make a reduction to a symbolic representation of the attractor K, which we will denote by Δ. Since this symbolic construction will later be required in §4, we establish this concept in a somewhat more general context.

3.2. Metric trees. First, fix a reference set Ω and write $\mathcal{T}_0 = \{\Omega\}$. Let $\{\mathcal{T}_k\}_{k=1}^{\infty}$ be a sequence of partitions of Ω so that $\mathcal{T}_{k+1}$ is a refinement of the partition $\mathcal{T}_k$. For each $Q \in \mathcal{T}_k$ with $k \in \mathbb{N}$, there is a unique *parent* $\widehat{Q} \in \mathcal{T}_{k-1}$ with $Q \subset \widehat{Q}$. Suppose that for any $\gamma_1 \neq \gamma_2 \in \Omega$ there is a $k \in \mathbb{N}$ such that there are $Q_1 \neq Q_2 \in \mathcal{T}_k$ so that $\gamma_1 \in Q_1$ and $\gamma_2 \in Q_2$. We call such a family $\{\mathcal{T}_k\}_{k=0}^{\infty}$ a *tree*, and write $\mathcal{T} = \bigcup_{k=0}^{\infty} \mathcal{T}_k$.

Now, suppose that there is a function $\rho\colon \mathcal{T} \to (0, \infty)$ which satisfies

1. $0 < \rho(Q) < \rho(\widehat{Q})$, and
2. there is a sequence $(r_k)_{k=1}^{\infty}$ converging to zero from above such that $\rho(Q) \leq r_k$ for all $Q \in \mathcal{T}_k$.

The function ρ induces a metric d on the space Ω by the rule

$$d(\gamma_1, \gamma_2) = \inf\{\rho(Q) : Q \in \mathcal{T} \text{ and } \{\gamma_1, \gamma_2\} \subset Q\}.$$

In particular, $\operatorname{diam}(Q) = \rho(Q)$ with respect to the metric d. We then refer to the data $(\Omega, \{\mathcal{T}_k\}_{k=0}^{\infty}, \rho)$ as a *metric tree*.

We say that a subset $\mathcal{A} \subset \mathcal{T}$ is a *section* if $Q_1 \cap Q_2 = \emptyset$ whenever $Q_1, Q_2 \in \mathcal{A}$ with $Q_1 \neq Q_2$. If $\bigcup_{Q \in \mathcal{A}} Q = Q_0$, we say that $\mathcal{A}$ is a *section relative to* Q_0. Note that sections are necessarily countable and, for example, each $\mathcal{T}_k$ for $k \in \mathbb{N} \cup \{0\}$ is a section relative to Ω. The set of sections is equipped with a partial order $\mathcal{A}_1 \preccurlyeq \mathcal{A}_2$ if for each $Q_1 \in \mathcal{A}_1$ there is a $Q_2 \in \mathcal{A}_2$ such that $Q_1 \subset Q_2$. This partial order is equipped with a *join*, that is, for finite family of sections $\mathcal{A}_1, \ldots, \mathcal{A}_m$, there is a unique section $\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m$ such that

$$\mathcal{A}_i \preccurlyeq \mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m \qquad \text{for all} \qquad i = 1, \ldots, m$$

and moreover $\mathcal{A}_1 \vee \cdots \vee \mathcal{A}_m$ is minimal with respect to $\preccurlyeq$ with this property.

A metric tree is equipped with a natural family of sections relative to Ω which respect the geometry of the metric d. We define

$$\mathcal{T}(r) = \{Q \in \mathcal{T} : \rho(Q) \leq r < \rho(\widehat{Q})\}$$

where, abusing notation, we write $\rho(\widehat{\Omega}) = \infty$. Property 1 above ensures that this is indeed a section and property 2 ensures that $\mathcal{T}_k \preccurlyeq \mathcal{T}(r)$ for all k sufficiently large.

3.3. Reduction to symbolic representation. Now that we have defined the metric tree, we introduce a symbolic representation of the set K. Let $\Delta = \prod_{n=1}^{\infty} \mathcal{J}_n$. For $(j_1, \ldots, j_n) \in \mathcal{J}_1 \times \cdots \times \mathcal{J}_n$, we denote the *cylinder*

$$[j_1, \ldots, j_n] = \{j_1\} \times \cdots \times \{j_n\} \times \prod_{k=n+1}^{\infty} \mathcal{J}_k.$$

We associate with this cylinder the valuation $\rho([j_1, \ldots, j_n]) = r_{1,j_1} \cdots r_{n,j_n}$. Let $\mathcal{T}_n$ denote the set of all cylinders corresponding to finite sequences in $\mathcal{J}_1 \times \cdots \times \mathcal{J}_n$. It is clear that this sequence of partitions, equipped with the valuation ρ (recalling the non-degeneracy assumption (3.1)), induces the structure of a metric tree on Δ.

We also define a natural projection $\pi\colon \Delta \to K$ by

$$\{\pi((j_n)_{n=1}^\infty)\} = \bigcap_{n=1}^\infty S_{1,j_1} \circ \cdots \circ S_{n,j_n}(X).$$

Again, this map is well-defined by (3.1). A direct argument shows that π is Lipschitz.

We now prove that $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}} \Delta$. The open set condition ensures that the only work in this result is to handle the mild overlaps which occur from adjacent rectangles. In fact, our result will follow from the following standard elementary lemma for metric spaces which are "almost bi-Lipschitz equivalent".

Lemma 3.2. *Let (X, d_1) and (Y, d_2) be non-empty bounded metric spaces and suppose there is a function $f\colon X \to Y$ and constants $M \in \mathbb{N}$ and $c > 0$ so that for any $0 < r < 1$,*

(i) $\operatorname{diam}(f(B(x, r))) \leq cr$ for all $x \in X$; and

(ii) for every $y \in Y$ there are $x_1, \ldots, x_M \in X$ such that $B(y, r) \subset \bigcup_{i=1}^M f(B(x_i, r))$.

Then $\dim_{\mathrm{A}} X = \dim_{\mathrm{A}} Y$.

Proof. Without loss of generality, we may assume that $c \geq 1$. Throughout the proof, let $\epsilon > 0$ and $0 < r \leq R < 1$ be arbitrary. First, let $x \in X$ be arbitrary and, writing, $N = N_r(f(B(x, R)))$, get $y_1, \ldots, y_N \in Y$ so that $f(B(x, R)) \subset \bigcup_{i=1}^N B(y_i, r)$. Since $\operatorname{diam} f(B(x, R)) \leq cR$,

$$N \lesssim_\epsilon \left(\frac{cR}{r}\right)^{\dim_{\mathrm{A}} Y + \epsilon} \lesssim \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} Y + \epsilon}.$$

Moreover, for each $i = 1, \ldots, N$, there are $x_{i,1}, \ldots, x_{i,M} \in X$ such that $B(y_i, r) \subset \bigcup_{j=1}^M f(B(x_{i,j}, r))$. Thus since $\{B(x_{i,j}, r) : i = 1, \ldots, N \text{ and } j = 1, \ldots, M\}$ is a cover for $B(x, R)$, it follows that

$$N_r(B(x, R)) \leq NM \lesssim_\epsilon \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} Y + \epsilon}.$$

Since $\epsilon > 0$ and $0 < r \leq R < 1$ are arbitrary, we see that $\dim_{\mathrm{A}} X \leq \dim_{\mathrm{A}} Y$.

Conversely, let $y \in Y$ be arbitrary and get $x_1, \ldots, x_M \in X$ such that $B(y, R) \subset \bigcup_{i=1}^M f(B(x_i, R))$. Moreover, for each $i = 1, \ldots, M$, writing $N_i = N_{c^{-1}r}(B(x_i, R))$, there are $x_{i,1}, \ldots, x_{i,N_i}$ where $B(x_i, R) \subset \bigcup_{j=1}^{N_i} B(x_{i,j}, c^{-1}r)$ and

$$N_i \lesssim_\epsilon \left(\frac{cR}{r}\right)^{\dim_{\mathrm{A}} X + \epsilon} \lesssim \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} X + \epsilon}.$$

Thus since $\{f(B(x_{i,j}, c^{-1}r)) : i = 1, \ldots, M \text{ and } j = 1, \ldots, N_i\}$ is a cover for $B(y, R)$ with $\operatorname{diam} f(B(x_{i,j}, c^{-1}r)) \leq r$, it follows that

$$N_r(B(y, R)) \lesssim_\epsilon \sum_{i=1}^M \sum_{j=1}^{N_i} \lesssim_\epsilon \left(\frac{R}{r}\right)^{\dim_{\mathrm{A}} X + \epsilon}.$$

Again since $\epsilon > 0$ and $0 < r \leq R < 1$ are arbitrary, we get $\dim_{\mathrm{A}} Y \leq \dim_{\mathrm{A}} X$, completing the proof. $\square$

We now obtain our result on the Assouad dimension as a direct corollary.

Corollary 3.3. *Let $\{\Phi_n\}_{n=1}^\infty$ be a sequence of self-similar IFSs with associated non-autonomous self-similar set K and metric tree Δ. Suppose the IFS also satisfies the open set condition and has uniformly bounded contractions. Then $\dim_A K = \dim_A \Delta$.*

Proof. Let $0 < r < 1$. First, recall that the map $\pi\colon \Delta \to K$ is Lipschitz. Moreover, if $[i_1, \ldots, i_m], [j_1, \ldots, j_\ell] \in \Delta(r)$, then

$$S_{1,i_1} \circ \cdots \circ S_{m,i_m}(U) \cap S_{1,j_1} \circ \cdots \circ S_{m,j_\ell}(U) = \emptyset$$

and by the uniformly bounded contraction assumption,

$$\operatorname{Leb}\left(S_{1,i_1} \circ \cdots \circ S_{m,i_m}(U)\right) \approx \operatorname{Leb}\left(S_{1,j_1} \circ \cdots \circ S_{\ell,j_\ell}(U)\right) \approx r^d.$$

But for $x \in K$, $\operatorname{Leb}(B(x,r)) \approx r^d$. Thus there is a constant $M \in \mathbb{N}$ not depending on r so that if $x \in K$ is arbitrary, there are cylinders $I_1, \ldots, I_M \in \Delta(r)$ so that $B(x,r) \subset \pi(I_1) \cup \cdots \cup \pi(I_M)$ so that each $I_j \in \Delta(r)$ and therefore $\operatorname{diam} I_j \leq r$. Thus the conditions for Lemma 3.2 are satisfied and $\dim_A K = \dim_A \Delta$. □

3.4. Regularity properties of Assouad dimension. In this section, we establish two regularity properties related to the Assouad dimension.

Lemma 3.4. *Let $A = \mathbb{R}^+$ or $A = \{\kappa_0 n : n \in \mathbb{N}\}$ for some $\kappa_0 > 0$. Suppose $f\colon A \times A \to \{-\infty\} \cup \mathbb{R}$ is any function satisfying the following two assumptions:*

(i) f is bounded from above.

(ii) For all $x, y, z \in A$,

$$f(x, y+z) \leq \frac{y \cdot f(x,y) + z \cdot f(x+y, z)}{y+z}.$$

Then

$$\begin{aligned} \beta &:= \limsup_{y\to\infty} \limsup_{x\to\infty} f(x,y) \\ &= \lim_{y\to\infty} \limsup_{x\to\infty} f(x,y) \\ &= \lim_{y\to\infty} \sup_{x\in A} f(x,y) \\ &= \inf_{y\in A} \limsup_{x\to\infty} f(x,y). \end{aligned}$$

Moreover, if $B \subset A$ is of the form $B = \{\kappa n : n \in \mathbb{N}\}$ for some $\kappa > 0$, then

$$\beta = \lim_{\substack{y\to\infty \\ y\in B}} \sup_{x\in B} f(x,y).$$

Proof. We assume that $\beta > -\infty$: the proof for $\beta = -\infty$ is similar (and substantially easier). Let $C \in \mathbb{R}$ be such that $f(x,y) \leq C$ for all $(x,y) \in A \times A$. Note that applying (ii) inductively, we obtain for any $\{y_i\}_{i=1}^\ell \subset A$ and $y \in A$

$$f\left(y, \textstyle\sum_{i=1}^\ell y_i\right) \leq \frac{\sum_{i=1}^\ell y_i f\left(y + \sum_{j=1}^{i-1} y_j, y_i\right)}{\sum_{i=1}^\ell y_i}. \tag{3.2}$$

We take the empty sum to be 0.

We first show that the limit defining β exists. Write $h(y) = y \cdot \limsup_{x\to\infty} f(x, y)$. Applying (3.2),

$$\begin{aligned} h(y_1 + y_2) &= (y_1 + y_2) \limsup_{x\to\infty} f(x, y_1 + y_2) \\ &\leq (y_1 + y_2) \limsup_{x\to\infty} \frac{y_1 f(x, y_1) + y_2 f(x + y_1, y_2)}{y_1 + y_2} \\ &\leq h(y_1) + h(y_2). \end{aligned}$$

Therefore the function $h\colon A \to \mathbb{R}$ is subadditive, so the limit $\lim_{y\to\infty} h(y)/y$ exists and is equal to $\inf_{y\in A} h(y)/y$. Note that the same argument applies with a supremum in place of the limit supremum.

We next show for each $\epsilon > 0$ and all y sufficiently large depending on ϵ and all $x \in A$,

$$f(x, y) \leq \beta + 3\epsilon. \tag{3.3}$$

By the definition of β, there are y_0 and K so that for all $x \geq K$, $f(x, y_0) \leq \beta+\epsilon$. Now let $y \in A$ be arbitrary and write $y = \ell y_0 + t$ for some $\ell \in \mathbb{N}\cup\{0\}$ and $0 < t \leq y_0$. Applying (3.2), there is some $M \in A$ (depending only on y_0 and C so that for all $y \geq M$,

$$\begin{aligned} f(x, y) &\leq \frac{y_0 \sum_{i=0}^{\ell-1} f(x + iy_0, y_0) + t f(x + \ell y_0, t)}{y} \\ &\leq \frac{\ell y_0}{y}(\beta + \epsilon) + \frac{Ct}{y} \leq \beta + 2\epsilon. \end{aligned}$$

Now let $x \in (0, K)$ and $y \geq M + K$, and set $t = K - x$. Again applying (3.2),

$$\begin{aligned} f(x, y) &\leq \frac{t f(x, t) + (y - t) f(K, y - t)}{y} \\ &\leq \frac{t}{y}C + \frac{y - t}{y}(\beta + 2\epsilon) \leq \beta + 3\epsilon \end{aligned}$$

for all y sufficiently large since $t \leq K$. This proves (3.3).

Finally, suppose $B \subset A$ is of the form $B = \{\kappa n : n \in \mathbb{N}\}$ for some $\kappa > 0$. First, note that since $B \subset A$,

$$\beta \geq \lim_{\substack{y\to\infty \\ y\in B}} \sup_{x\in B} f(x, y)$$

and moreover the limit exists as proven above. Conversely, let $(x, y) \in A \times A$ be arbitrary with $y \geq 2\kappa$ and get $(x_0, y_0) \in B \times B$ and $(t_x, t_y) \in A \times A$ with $t_x, t_y \leq \kappa$ such that $x = x_0 + t_x$ and $y - t_x = y_0 + t_y$. Then applying (ii) twice,

$$\begin{aligned} f(x_0, y_0) &\geq \frac{y_0 + t}{y_0} \cdot f(x, y - t_x) - \frac{t}{y_0} f(x + y_0, t) \\ &\geq f(x + t_x, y - t_x) - \frac{C\kappa}{y - 2\kappa} \end{aligned}$$

$$\geq \frac{y}{y-t_x} \cdot f(x,y) - \frac{t_x}{y-t_x} f(x, y-t_x) - \frac{C\kappa}{y-2\kappa}$$
$$\geq f(x,y) - 2\frac{C\kappa}{y-2\kappa}.$$

Therefore since C and κ are fixed and $y_0 \geq y - 2\kappa$,

$$\limsup_{\substack{y_0 \to \infty \\ y_0 \in B}} \sup_{x \in B} f(x_0, y_0) \geq \limsup_{y \to \infty} \sup_{x \in A} f(x,y) = \beta$$

as required. □

As an application, we can use this subadditivity result to obtain a nice reformulation of the Assouad dimension of an arbitrary set. Let X be a compact doubling metric space and for $\delta \in (0,1)$ and $r \in (0,1)$, write

$$\psi(r, \delta) = \sup_{x \in X} N_{r\delta}\big(B(x,r) \cap K\big)$$

and then set

$$\Psi(r,\delta) = \frac{\log \psi(r,\delta)}{\log(1/\delta)}.$$

One can think of $\Psi(r,\delta)$ is the best guess for the Assouad dimension of X at scales $0 < r\delta < \delta < 1$. This heuristic is made precise in the following result.

Corollary 3.5. *Let X be a compact doubling metric space. Then*

$$\dim_{\mathrm{A}} X = \limsup_{\delta \to 0} \limsup_{r \to 0} \Psi(r,\delta) = \lim_{\delta \to 0} \sup_{r \in (0,1)} \Psi(r,\delta). \tag{3.4}$$

Proof. Since X is doubling, there is an $M \geq 0$ so that $\Psi(r,\delta) \in [0, M]$. Moreover, given $r, \delta_1, \delta_2 \in (0,1)$, by covering balls $B(x, r\delta_1)$ by balls of radius $r\delta_1\delta_2$,

$$\psi(r, \delta_1\delta_2) \leq \psi(r,\delta_1)\psi(r\delta_1, \delta_2)$$

and therefore

$$\begin{aligned}\Psi(r,\delta_1\delta_2) &= \frac{\log \psi(r,\delta_1\delta_2)}{\log(1/\delta_1\delta_2)} \\ &\leq \frac{\log \psi(r,\delta_1) + \log \psi(r\delta_1,\delta_2)}{\log(1/\delta_1\delta_2)} \\ &= \frac{\log(1/\delta_1)\Psi(r,\delta_1) + \log(1/\delta_2)\Psi(r\delta_1,\delta_2)}{\log(1/\delta_1) + \log(1/\delta_2)}.\end{aligned}$$

Thus with the change of coordinate $g(x,y) = (e^{-x}, e^{-y})$, the second equality in (3.4) follows by applying Lemma 3.4 to the function $\Psi \circ g$.

To see the first equality in (3.4), it is a direct consequence of the definition of the Assouad dimension that

$$\limsup_{\delta \to 0} \limsup_{r \to 0} \Psi(r,\delta) \leq \dim_{\mathrm{A}} K$$

and that there are sequences $(\delta_n)_{n=1}^\infty$ and $(r_n)_{n=1}^\infty$ with $\lim_{n\to\infty}\delta_n = 0$ such that

$$\lim_{\delta\to 0}\sup_{r\in(0,1)}\Psi(r,\delta) \geq \limsup_{n\to\infty}\Psi(r_n,\delta_n) \geq \dim_{\mathrm{A}} K,$$

as required. □

Finally, we prove that in the definition of the Assouad dimension one may replace the exponent associated to localized coverings of balls of the same size by an exponent coming from localized packings of balls which may have different sizes. This will be useful since the natural covers appearing from the symbolic representation of K consist of cylinders which may have very non-uniform diameters when indexed by length. First, for a metric space X, $x \in X$, and $R \in (0,1)$, denote the family of all localized centred packings by

$$\operatorname{pack}(X,x,R) = \left\{ \{B(x_i,r_i)\}_{i=1}^\infty : \begin{array}{c} 0 < r_i \leq R, x_i \in X, B(x_i,r_i) \subset B(x,R), \\ B(x_i,r_i)\cap B(x_j,r_j) = \emptyset \text{ for all } i \neq j \end{array} \right\}.$$

In our proof, we will also require the Assouad dimension of a measure. Given a compact doubling metric space X and a Borel measure μ with $\operatorname{supp}\mu = X$, the *Assouad dimension of* μ is given by

$$\dim_{\mathrm{A}}\mu = \inf\Big\{\alpha \geq 0 : \forall x \in X\, \forall 0 < r \leq R < \operatorname{diam} X \\ \frac{\mu(B(x,R))}{\mu(B(x,r))} \lesssim_\alpha \left(\frac{R}{r}\right)^\alpha\Big\}.$$

The main result of [VK88] (the original Russian version can be found in [VK87]) is that for a compact doubling metric space X,

$$\dim_{\mathrm{A}} X = \inf\{\dim_{\mathrm{A}}\mu : \operatorname{supp}\mu = X\}.$$

In the following result, we observe that the existence of good measures provides a convenient way to control the localized disk packing exponent.

Proposition 3.6. *Let X be a bounded metric space. Then*

$$\dim_{\mathrm{A}} X = \inf\Big\{\alpha : \forall 0 < R < 1\, \forall x \in X\, \forall \{B(x_i,r_i)\}_{i=1}^\infty \in \operatorname{pack}(X,x,R) \\ \sum_{i=1}^\infty r_i^\alpha \lesssim_\alpha R^\alpha\Big\}.$$

Proof. That

$$\dim_{\mathrm{A}} X \leq \inf\Big\{\alpha : \forall 0 < R < 1\, \forall x \in X\, \forall \{B(x_i,r_i)\}_{i=1}^\infty \in \operatorname{pack}(X,x,R) \\ \sum_{i=1}^\infty r_i^\alpha \lesssim_\alpha R^\alpha\Big\}$$

is immediate by specializing to packings with $r_i = r$ for some $0 < r \leq R$, using the equivalence (up to a constant factor) of covering and packing counts.

Now to show the lower bound, if X is not doubling, then $\dim_{\mathrm{A}} X = \infty$ and the result is trivial. Otherwise, by passing to the completion (which does not change the value of the Assouad dimension) and recalling that a bounded doubling metric space is totally bounded, we may assume that X is also compact. Thus let $\alpha > \dim_{\mathrm{A}} X$ be arbitrary. By [VK88, Theorem 1], there is a probability measure μ with $\operatorname{supp} \mu = X$ and $\dim_{\mathrm{A}} \mu < \alpha$. Then for any $0 < R < 1$, $x \in X$, and $\{B(x_i, r_i)\}_{i=1}^{\infty} \in \operatorname{pack}(X, x, R)$, by disjointness,

$$\mu\big(B(x,R)\big) \geq \sum_{i=1}^{\infty} \mu\big(B(x_i, r_i)\big) \gtrsim \mu\big(B(x,R)\big) \sum_{i=1}^{\infty} \Big(\frac{r_i}{R}\Big)^{\alpha}.$$

Therefore,

$$\sum_{i=1}^{\infty} r_i^{\alpha} \lesssim R^{\alpha}$$

which, since $\alpha > \dim_{\mathrm{A}} X$ was arbitrary, yields the claimed result. □

3.5. Proof of the Assouad dimension formula. We can now state and prove the desired formula for the Assouad dimension of the non-autonomous self-similar set K. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$ be arbitrary, and let $\theta(n,m)$ denote the unique value satisfying the equation

$$\sum_{j_1 \in \mathcal{J}_{n+1}} \cdots \sum_{j_m \in \mathcal{J}_{n+m}} \prod_{k=1}^{m} r_{n+k, j_k}^{\theta(n,m)} = 1.$$

Note that $\theta(n,m)$ is precisely the similarity dimension of the IFS

$$\Phi_{n+1} \circ \cdots \circ \Phi_{n+m} = \{f_1 \circ \cdots \circ f_m : f_i \in \Phi_{n+i}\}.$$

We establish the following formula for the Assouad dimension of K.

Theorem 3.7. *Let $(\Phi_n)_{n=1}^{\infty}$ be a non-autonomous IFS satisfying the open set condition and with uniformly bounded contraction ratios. Denote the associated non-autonomous self-similar set by K. Then*

$$\dim_{\mathrm{A}} K = \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \theta(n,m). \tag{3.5}$$

Proof. Let us first show that the limit in (3.5) exists by verifying that the function $\theta(n,m)$ satisfies the assumptions of Lemma 3.4 with $A = \mathbb{N}$. First, $\theta(n,m) \in [0,d]$ since $\#\mathcal{J}_n \geq 1$ for all $n \in \mathbb{N}$ and the open set condition along with scaling properties of d-dimensional Lebesgue measure forces $\sum_{j \in \mathcal{J}_n} r_{n,j}^d \leq 1$. Thus it remains to verify assumption (ii) in Lemma 3.4. Let $n, m_1, m_2 \in \mathbb{N}$ be arbitrary. Recalling the definitions of $\theta(n,m_1)$ and $\theta(n+m_1,m_2)$, by Hölder's inequality with exponents $(m_1+m_2)/m_1$ and $(m_1+m_2)/m_2$,

$$\begin{aligned} 1 &= \sum_{j_1\in\mathcal{J}_{n+1}}\cdots\sum_{j_{m_1+m_2}\in\mathcal{J}_{n+m_1+m_2}}\left(\prod_{k=1}^{m_1} r_{n+k,j_k}\right)^{\theta(n,m_1)}\left(\prod_{k=m_1+1}^{m_1+m_2} r_{n+k,j_k}\right)^{\theta(n+m_1,m_2)} \\ &\geq \sum_{j_1\in\mathcal{J}_{n+1}}\cdots\sum_{j_{m_1+m_2}\in\mathcal{J}_{n+m_1+m_2}}\left(\prod_{k=1}^{m_1+m_2} r_{n+k,j_k}\right)^{\frac{m_1\theta(n,m_1)+m_2\theta(n+m_1,m_2)}{m_1+m_2}}. \end{aligned}$$

But $\theta(n,m_1+m_2)$ is the unique value satisfying $\phi(\theta(n,m_1+m_2))=1$, where

$$\phi(s)=\sum_{j_1\in\mathcal{J}_{n+1}}\cdots\sum_{j_{m_1+m_2}\in\mathcal{J}_{n+m_1+m_2}}\left(\prod_{k=1}^{m_1+m_2} r_{n+k,j_k}\right)^{s}$$

is monotonically decreasing in s, yielding the desired inequality. In particular, the limit in (3.5) exists.

Let us now verify the formula. First, recall from Corollary 3.3 that $\dim_{\mathrm{A}} K = \dim_{\mathrm{A}} \Delta$. Let $\epsilon > 0$ be fixed and let M be sufficiently large so that for all $n\in\mathbb{N}$ and $m\geq M$,

$$|\theta(n,m)-s|\leq\epsilon \qquad \text{where} \qquad s=\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\theta(n,m).$$

Now fix a cylinder $[j_1,\ldots,j_n]\subset\Delta$ for some $(j_1,\ldots,j_n)\in\mathcal{J}_1\times\cdots\times\mathcal{J}_n$ and write $R=\operatorname{diam}([j_1,\ldots,j_n])=r_{1,j_1}\cdots r_{n,j_n}$. Note that if $m\geq M$, by definition of $\theta(n,m)$

$$\sum_{j_{n+1}\in\mathcal{J}_{n+1}}\cdots\sum_{j_{n+m}\in\mathcal{J}_{n+m}}\prod_{k=1}^{n+m} r_{k,j_k}^{\theta(n,m)}=R^{\theta(n,m)}.$$

But the family of cylinders

$$\left\{[j_1,\ldots,j_{n+m}]:(j_{n+1},\ldots,j_{n+m})\in\mathcal{J}_{n+1}\times\cdots\times\mathcal{J}_{n+m}\right\}$$

forms a packing of $B(x,R)$. Thus since $m\geq M$ is arbitrary, by Proposition 3.6, $\dim_{\mathrm{A}} K\geq s-\epsilon$.

Conversely, let us upper bound $\dim_{\mathrm{A}} K$. Recall that $\epsilon>0$ is fixed as above and let $m\geq M$ be fixed. Now let $0<r\leq R<1$ and fix a ball $B(x,R)\subset\Delta$. By definition of the metric on Δ, $B(x,R)=[j_1,\ldots,j_n]$ where $r_{1,j_1}\cdots r_{n,j_n}\leq R$. We inductively build a sequence of covers $(\mathcal{B}_k)_{k=1}^{\infty}$ for $B(x,R)$ such that each $\mathcal{B}_k$ is composed only of cylinder sets and

$$\sum_{[i_1,\ldots,i_\ell]\in\mathcal{B}_k}(r_{1,i_1}\cdots r_{\ell,i_\ell})^{s+\epsilon}\leq R^{s+\epsilon}. \tag{3.6}$$

and

$$r_{1,i_1}\cdots r_{\ell,i_\ell}\geq r\cdot r_{\mathrm{min}}^{m} \qquad \text{for all} \qquad [i_1,\ldots,i_\ell]\in\mathcal{B}_k. \tag{3.7}$$

Begin with $\mathcal{B}_1=\{[j_1,\ldots,j_n]\}$, which clearly satisfies the requirements.

Now suppose we have constructed $\mathcal{B}_k$ for some $k \in \mathbb{N}$. Let $[i_1, \ldots, i_\ell] \in \mathcal{B}_k$ be an arbitrary cylinder set. If $r_{1,i_1} \cdots r_{\ell,i_\ell} \leq r$, do nothing; this guarantees that (3.7) holds. Otherwise, replace the cylinder $[i_1, \ldots, i_\ell]$ with the family of cylinders

$$\{[i_1, \ldots, i_\ell, j_1, \ldots, j_m] : (j_1, \ldots, j_m) \in \mathcal{J}_{\ell+1} \times \cdots \times \mathcal{J}_{\ell+m}\}.$$

The choice of $m \geq M$ and the definition of $\theta(\ell, m)$ ensures that (3.6) holds.

Repeat this process until every cylinder in $\mathcal{B}_k$ has diameter $\leq r$. That this process terminates at a finite level k is guaranteed by (3.1). Thus replacing each cylinder $[i_1, \ldots, i_\ell]$ with a ball $B(x_{i_1,\ldots,i_\ell}, r)$ for some $x_{i_1,\ldots,i_\ell} \in [i_1, \ldots, i_\ell]$, by (3.6) and (3.7) the corresponding cover has

$$\sum_{[i_1,\ldots,i_\ell] \in \mathcal{B}_k} r^{s+\epsilon} \leq r_{\min}^{-m(s+\epsilon)} \sum_{[i_1,\ldots,i_\ell] \in \mathcal{B}_k} (r_{1,i_1} \cdots r_{\ell,i_\ell})^{s+\epsilon} \lesssim R^{s+\epsilon}$$

which guarantees that $\dim_{\mathrm{A}} K \leq s + \epsilon$, as claimed. □

4. TANGENT STRUCTURE AND DIMENSION OF GATZOURAS–LALLEY CARPETS

In this section, we introduce the definitions of Gatzouras–Lalley and Barański carpets and prove our main results on tangents and pointwise Assouad dimension of Gatzouras–Lalley carpets.

4.1. Gatzouras–Lalley and Barański carpets.

4.1.1. Defining the maps. Fix an index set $\mathcal{I}$ with $\#\mathcal{I} \geq 2$, and for $j = 1, 2$ fix contraction ratios $(\beta_{i,j})_{i \in \mathcal{I}}$ in $(0, 1)$ and translations $(d_{i,j})_{i \in \mathcal{I}} \in \mathbb{R}$. We then call the IFS $\{T_i\}_{i \in \mathcal{I}}$ *diagonal* when

$$T_i(x_1, x_2) = (\beta_{i,1} x_1 + d_{i,1}, \beta_{i,2} x_2 + d_{i,2}) \text{ for each } i \in \mathcal{I}.$$

Let η_j denote the orthogonal projection onto the j^{th} coordinate axis, i.e. $\eta_j(x_1, x_2) = x_j$. We denote by $\Lambda_j = \{S_{i,j}\}_{i \in \mathcal{I}}$ the *projected systems*, where $\eta_j \circ T_j = S_{i,j} \circ \eta_j$. We will often write $\eta = \eta_1$ to denote simply the projection onto the first coordinate axis. Of course, $S_{i,j}(x) = \beta_{i,j} x + d_{i,j}$ are iterated function systems of similarities.

Let $\mathcal{I}^* = \bigcup_{n=0}^{\infty} \mathcal{I}^n$, and for $\mathtt{i} = (i_1, \ldots, i_n) \in \mathcal{I}^*$ and $j = 1, 2$, write

$$T_{\mathtt{i}} = T_{i_1} \circ \cdots \circ T_{i_n},$$
$$S_{\mathtt{i},j} = S_{i_1,j} \circ \cdots \circ S_{i_n,j}$$

and

$$p_{\mathtt{i}} = p_{i_1} \cdots p_{i_n},$$
$$\beta_{\mathtt{i},j} = \beta_{i_1,j} \cdots \beta_{i_n,j}.$$

For $n \in \mathbb{N}$ and $\gamma \in \Omega := \mathcal{I}^{\mathbb{N}}$, we write $\gamma|_n$ to denote the unique prefix of γ in $\mathcal{I}^n$.

Now η_j induces an equivalence relation $\sim_j$ on $\mathcal{I}$ where $i \sim i'$ if $S_{i,j} = S_{i',j}$. Let $\eta_j \colon \mathcal{I} \to \mathcal{I}/\sim_j$ denote the natural projection. Intuitively, $\eta_j(i)$ is the set of indices which lie in the same column or row as the index i. Then η_j extends naturally to a map on Ω by $\eta_j((i_n)_{n=1}^\infty) = (\eta_j(i_n))_{n=1}^\infty \subset \eta_j(\mathcal{I})^*$; and similar extends to a map on $\mathcal{I}^*$. For notational clarity, we will refer to words in $\mathcal{I}^*$ using upright indices, such as $\mathtt{i}$, and words in $\eta_j(\mathcal{I}^*)$ using their underlined variants, such as $\underline{\mathtt{i}}$. Note that if $\mathtt{i} \sim_j \mathtt{j}$, then and $S_{\mathtt{i},j} = S_{\mathtt{j},j}$. In particularly, we may unambiguously write $S_{\underline{\mathtt{i}},j}$ and $\beta_{\underline{\mathtt{i}},j}$ for $\underline{\mathtt{i}} \in \eta_j(\mathcal{I}^*)$.

Associated with the IFS $\{T_i\}_{i\in\mathcal{I}}$ is a unique non-empty compact *attractor* K, satisfying $K = \bigcup_{i\in\mathcal{I}} T_i(K)$. Note that the projected IFS $\{S_{i,j}\}_{i\in\mathcal{I}}$ has attractor $K_j = \eta_j(K)$ for $j = 1, 2$. Recalling that $\Omega = \mathcal{I}^{\mathbb{N}}$, let $\pi \colon \Omega \to K$ denote the continuous map

$$\bigl\{\pi\bigl((i_n)_{n=1}^\infty\bigr)\bigr\} = \lim_{n\to\infty} S_{i_1} \circ \cdots \circ S_{i_n}(K).$$

Without loss of generality, and for the remainder of this document, we will assume that $K \subset [0,1]^2$. We can now introduce our two primary classes of self-affine sets.

Definition 4.1. We say that the carpet is of type *Gatzouras–Lalley* if:

1. $T_i((0,1)^2) \cap T_j((0,1)^2) = \emptyset$ for all $i \neq j$,
2. either $S_{i,1}((0,1)) = S_{(j,1)}((0,1))$ or $S_{i,1}((0,1)) \cap S_{j,1}((0,1)) = \emptyset$ for all i, j, and
3. $\beta_{i,1} > \beta_{i,2}$ for all $i \in \mathcal{I}$;

and type *Barański* if:

1. $T_i((0,1)^2) \cap T_j((0,1)^2) = \emptyset$ for all $i \neq j$, and
2. either $S_{i,\ell}((0,1)) = S_{(j,\ell)}([0,1])$ or $S_{i,\ell}((0,1)) \cap S_{j,\ell}((0,1)) = \emptyset$ for all i, j and $\ell = 1, 2$.

Moreover, we say that an IFS $\{f_i\}_{i\in\mathcal{I}}$ with attractor K satisfies the *strong separation condition* (or SSC for short) if $f_i(K) \cap f_j(K) = \emptyset$ for all $i \neq j \in \mathcal{I}$.

4.1.2. Dimensions of Gatzouras–Lalley carpets. To conclude this section, we recall some standard results on the dimensions of Gatzouras–Lalley carpets. We defer the corresponding results for Barański carpets to §5.1.

Before we do this, we first recall the notion of the lower dimension, which is in some sense dual to the definition of Assouad definition. Let $K \subset \mathbb{R}^d$ be compact. Then the *lower dimension* of K is given by

$$\dim_{\mathrm{L}} K = \sup\Bigl\{s : \exists C > 0\, \forall 0 < r \leq R < 1\, \forall x \in K \\ N_r(B(x,R) \cap K) \geq C\Bigl(\frac{R}{r}\Bigr)^s\Bigr\}.$$

In order to state our results on the Hausdorff dimensions, we must also introduce some notation for Bernoulli measures. Let $\mathcal{P}$ denote the collection of probability vectors on $\mathcal{I}$, i.e.

$$\mathcal{P} = \mathcal{P}(\mathcal{I}) \coloneqq \Bigl\{(p_i)_{i\in\mathcal{I}} : p_i \geq 0 \text{ for all } i \text{ and } \sum_{i\in\mathcal{I}} p_i = 1\Bigr\}.$$

Equip $\mathcal{P}$ with the topology inherited from $\mathbb{R}^{\mathcal{I}}$. Given $\boldsymbol{p} \in \mathcal{P}$, considering $\boldsymbol{p}$ as a probability measure on $\mathcal{I}$, we let $\boldsymbol{p}^{\mathbb{N}}$ denote the infinite product measure supported on Ω. We let $\mu_{\boldsymbol{p}} = \pi_* \boldsymbol{p}^{\mathbb{N}}$ denote the corresponding invariant measure on K, where π_* denotes the pushforward map. Note that the projections η_j also induce natural maps $\eta_j : \mathcal{P}(\mathcal{I}) \to \mathcal{P}(\eta_j(\mathcal{I}))$ by $\eta_j(\boldsymbol{p})_{\underline{\ell}} = \sum_{i \in \eta_j^{-1}(\underline{\ell})} \boldsymbol{p}_i$.

Given a probability vector $\boldsymbol{p} \in \mathcal{P}$, we write

$$H(\boldsymbol{p}) = -\sum_{i \in \mathcal{I}} p_i \log p_i \qquad \text{and} \qquad \chi_j(\boldsymbol{p}) = -\sum_{i \in \mathcal{I}} p_i \log \beta_{i,j}.$$

We now recall the main results of [LG92]—stated below in (i) and (ii)—as well as the result of [Mac11]—stated below in (iii). We also note that the same proof as given in [Mac11] (which is explained more precisely in [Fra14, Theorem 2.13]) gives the analogous result for the lower dimension.

Proposition 4.2 ([LG92; Mac11]). *Let K be a Gatzouras–Lalley carpet.*

(i) The Hausdorff dimension of K is given by

$$\dim_{\mathrm{H}} K = \sup_{\boldsymbol{p} \in \mathcal{P}} s(\boldsymbol{p})$$

where

$$s(\boldsymbol{p}) := \frac{H(\boldsymbol{p})}{\chi_2(\boldsymbol{p})} + \frac{H(\eta(\boldsymbol{p}))}{\chi_1(\boldsymbol{p}) - \chi_2(\boldsymbol{p})}.$$

Moreover, the supremum is always attained at an interior point of $\mathcal{P}$.

(ii) The box dimension of K exists and is given by the unique solution to

$$\sum_{i \in \mathcal{I}} \beta_{i,1}^{\dim_{\mathrm{B}} \eta(K)} \beta_{i,2}^{\dim_{\mathrm{B}} K - \dim_{\mathrm{B}} \eta(K)} = 1 \qquad \text{where} \qquad \sum_{\underline{j} \in \eta(\mathcal{I})} \beta_{\underline{j},1}^{\dim_{\mathrm{B}} \eta(K)} = 1.$$

(iii) The Assouad dimension of K is given by

$$\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + \max_{\underline{\ell} \in \eta(\mathcal{I})} t(\underline{\ell})$$

where $t(\underline{\ell})$ is defined as the unique solution to the equations

$$\sum_{j \in \eta^{-1}(\underline{\ell})} \beta_{j,2}^{t(\underline{\ell})} = 1.$$

Similarly, the lower dimension of K is given by

$$\dim_{\mathrm{L}} K = t + \min_{\underline{\ell} \in \eta(\mathcal{I})} t(\underline{\ell}).$$

4.1.3. Regular points and interior words. We conclude this section with the notion of a regular point and an interior word.

Definition 4.3. We say that a point $x \in K$ is *regular* if for each $r \in (0,1)$, there is an $\mathtt{i} \in \mathcal{I}^*$ with $\beta_{\mathtt{i},1} \lesssim r$ such that $B(\eta(x), r) \cap \eta(K) \subset S_{\mathtt{i},1}(K)$. Given $\mathtt{i} \in \mathcal{I}^*$, we say that $\mathtt{i}$ is an *interior word* if $S_{\mathtt{i},1}([0,1]) \subset (0,1)$. We let $\mathcal{B}_n \subset \mathcal{I}^n$ denote the set of interior words of length n.

The following lemma is standard. Here, and elsewhere, given an $n \in \mathbb{N}$ and $\mathcal{Y} \subset \mathcal{I}^n$, we write $\mathcal{Y}^{\mathbb{N}} \cong \Omega$ to denote the natural bijection. We will frequently abuse notation and interchangeably refer to elements in the subsystem or in the full system.

Lemma 4.4. *Let K be a Gatzouras–Lalley carpet.*
(i) If $\eta(K)$ satisfies the SSC, then each $x \in K$ is regular.
(ii) Suppose $\gamma \in \mathcal{B}_n^{\mathbb{N}} \cong \Omega$ for some $n \in \mathbb{N}$. Then $\pi(\gamma)$ is regular.

We can now guarantee the existence of large subsystems consisting only of regular points. This result is essentially [FJS10, Lemma 4.3].

Proposition 4.5 ([FJS10]). *Let K be a Gatzouras–Lalley carpet corresponding to the IFS $\{T_i\}_{i\in\mathcal{I}}$. Then for every $\epsilon > 0$, there is an $n \in \mathbb{N}$ and a family $\mathcal{J} \subset \mathcal{I}^n$ so that the IFS $\Lambda = \{T_{\mathtt{j}} : \mathtt{j} \in \mathcal{J}\}$ with attractor K_ϵ satisfies the following conditions:*
(i) each $\mathtt{i} \in \mathcal{J}$ is an interior word,
(ii) $\dim_{\mathrm{H}} K_\epsilon \geq \dim_{\mathrm{H}} K - \epsilon$,
(iii) $\dim_{\mathrm{B}} \eta(K_\epsilon) \geq \dim_{\mathrm{B}} \eta(K) - \epsilon$, and
(iv) there are $0 < \rho_2 < \rho_1 < 1$ so that $\beta_{\mathtt{i},1} = \rho_1$ and $\beta_{\mathtt{i},2} = \rho_2$ for all $\mathtt{i} \in \mathcal{I}$ and each column has the same number of maps.
In particular, each $x \in K_\epsilon$ is a regular point with respect to the IFS $\{T_i\}_{i\in\mathcal{I}}$ and $\dim_{\mathrm{A}} K_\epsilon = \dim_{\mathrm{H}} K_\epsilon = \dim_{\mathrm{L}} K_\epsilon$.

Proof. First, if K is contained in a vertical line, then K is the attractor of a self-similar IFS in $\mathbb{R}$ and the result is substantially easier. Now applying [FJS10, Lemma 4.3], there exists a family $\mathcal{J}_0 \subset \mathcal{I}^{n_0}$ with attractor K_0 satisfying conditions (ii), (iii), and (iv). By condition (iv), $t(\mathtt{i}) = t$ for all $\mathtt{i} \in \mathcal{J}_0$ and

$$\dim_{\mathrm{H}} K_0 = \dim_{\mathrm{B}} \eta(K) + t$$

and since K is not contained in a vertical line, we may assume that $\dim_{\mathrm{B}} \eta(K_0) > 0$.

Since $\eta(K_0)$ is the attractor of a self-similar IFS, iterating $\mathcal{J}_0$ if necessary and removing the maps in the first and last column, obtain a family $\mathcal{J} \subset \mathcal{J}_0^n$ with corresponding attractor K_ϵ such that $t(\mathtt{j}) = t$ for any $\mathtt{j} \in \mathcal{J}$, and $\dim_{\mathrm{B}} \eta(K_\epsilon) \geq \dim_{\mathrm{B}} \eta(K) - \epsilon$. Since words which correspond to rectangles that do not lie in the first or last column are necessarily interior words, combining this construction with Lemma 4.4 provides a family $\mathcal{J}$ satisfying the desired properties. $\square$

4.2. Approximate squares and symbolic slices. A common technique when studying invariant sets for iterated function systems on some index set $\mathcal{I}$ is to first reduce the problem to a symbolic problem on the coding space $\mathcal{I}^*$. However,

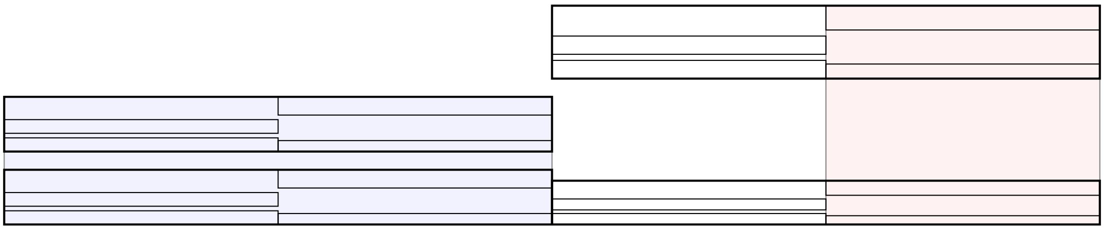


FIGURE 3. Two iterations of a Gatzouras–Lalley IFS within a cylinder, with a wide pseudo-cylinder in highlighted in blue and a tall pseudo-cylinder in red.

the main technical complexity in understanding the dimension theory Gatzouras–Lalley carpets, and more generally self-affine sets, is that the cylinder sets $T_{\mathtt{i}}(K)$ are often exponentially distorted rectangles. As a result, we will keep track of two symbolic systems simultaneously, which together will capture the geometry of the set K.

Fix a Gatzouras–Lalley IFS $\Lambda = \{T_i\}_{i \in \mathcal{I}}$. We first introduce some notation for handling cylinders. We then associate with the IFS Λ, and the related defining data that we introduced in §4.1, two important metric trees: first, the metric tree of *approximate squares*, and second the metric tree of *symbolic slices*.

First, recall that $\Omega = \mathcal{I}^{\mathbb{N}}$ is the space of infinite sequences on $\mathcal{I}$. Given $k \in \mathbb{N} \cup \{0\}$ and a word $\mathtt{i} \in \mathcal{I}^k$, we define the *cylinder* corresponding to $\mathtt{i}$ by

$$[\mathtt{i}] = \{\gamma \in \Omega : \gamma|_k = \mathtt{i}\}.$$

The family of cylinders $\{[\mathtt{i}] : \mathtt{i} \in \mathcal{I}^k\}_{k=0}^{\infty}$ defines a tree: we will often abuse notation and simply refer to $\{\mathcal{I}^k\}_{k=0}^{\infty}$ as a tree. We will associate with this tree a variety of metrics, such as those induced by the maps $\mathtt{i} \mapsto \beta_{\mathtt{i},j}$ for $j = 1, 2$. We will also use the same notation for the projected words $\{\eta(\mathcal{I}^k)\}_{k=0}^{\infty}$.

Next, we define the *metric tree of approximate squares*. Before we do this, we introduce the notion of a *pseudo-cylinder*. Suppose $\mathtt{i} \in \mathcal{I}^k$ and $\underline{\mathtt{j}} \in \eta(\mathcal{I}^\ell)$. We then write

$$P(\mathtt{i}, \underline{\mathtt{j}}) = \{\gamma = (i_n)_{n=1}^{\infty} \in \Omega : (i_1, \ldots, i_k) = \mathtt{i} \text{ and } \eta(i_{k+1}, \ldots, i_{k+\ell}) = \underline{\mathtt{j}}\}.$$

If in addition

$$\beta_{\mathtt{i}\underline{\mathtt{j}},1} \geq \beta_{\mathtt{i},2}, \tag{4.1}$$

we refer to the pseudo-cylinder as *wide*; otherwise, we refer to the pseudo-cylinder as *tall*. Note that map $(\mathtt{i}, \underline{\mathtt{j}}) \mapsto P(\mathtt{i}, \underline{\mathtt{j}})$ is injective. Another equivalent way to understand the pseudo-cylinder $P(\mathtt{i}, \underline{\mathtt{j}})$ is as a finite union of cylinders inside the cylinder $[\mathtt{i}]$, all of which lie inside the same column; that is,

$$P(\mathtt{i}, \underline{\mathtt{j}}) = \bigcup_{\mathtt{k} \in \eta^{-1}(\underline{\mathtt{j}})} [\mathtt{i}\mathtt{k}]. \tag{4.2}$$

We refer the reader to Figure 3 for a depiction of the definition of a pseudo-cylinder.

Now given an infinite word $\gamma \in \Omega$, let $L_k(\gamma)$ be the maximal integer so that

$$\beta_{\gamma_1,1} \cdots \beta_{\gamma_{L_k(\gamma)},1} \geq \beta_{\gamma_1,2} \cdots \beta_{\gamma_k,2}.$$

In other words, $L_k(\gamma)$ is chosen so that the level $L_k(\gamma)$ rectangle has approximately the same width as the height of the level k rectangle. Write $\gamma|_{L_k(\gamma)} = \mathtt{i}\mathtt{j}$ where $\mathtt{i} \in \mathcal{I}^k$. We then define the *approximate square* $Q_k(\gamma) \subset \Omega$ by

$$Q_k(\gamma) = P(\mathtt{i}, \eta(\mathtt{j})).$$

While different γ may define the same approximate square, the choice of $\mathtt{i}$ and $\eta(\mathtt{j})$ are unique. Note that the index $L_k(\gamma)$ is chosen precisely to be as large as possible while still satisfying the condition (4.1). In other words, one can think of the wide pseudo-cylinders as "interpolating" between the cylinder $P(\mathtt{i}, \varnothing) = [\mathtt{i}]$ and the approximate square $P(\mathtt{i}, \eta(\mathtt{j})) = Q_k(\gamma)$, with the approximate square lying precisely at the threshold at which the pseudo-cylinder becomes tall.

Of course, $Q_{k+1}(\gamma) \subset Q_k(\gamma)$ and moreover for any $\gamma, \gamma' \in \Omega$, either $Q_k(\gamma) = Q_k(\gamma')$ or $Q_k(\gamma) \cap Q_k(\gamma') = \emptyset$. Denote the set of all approximate squares by

$$\mathcal{S}_k = \{Q_k(\gamma) : \gamma \in \Omega\} \qquad \text{and} \qquad \mathcal{S} = \bigcup_{k=0}^{\infty} \mathcal{S}_k.$$

As discussed above, every approximate square is uniquely associated with a pair $(\mathtt{i}, \underline{\mathtt{j}})$, so we may therefore define a metric induced by $\rho(Q) = \beta_{\mathtt{i},2}$, which makes the collection of approximate squares into a metric tree.

To conclude this section, we define the *metric tree of symbolic slices*. Suppose we fix a word $\gamma \in \Omega$. The word $\gamma = (i_n)_{n=1}^{\infty}$ defines for each $n \in \mathbb{N}$ a self-similar IFS $\Phi_n = \{S_{j,2} : j \in \eta^{-1}(\eta(i_n))\}$. This IFS is precisely the IFS corresponding to the column containing the index i_n. Note that there are only finitely many possible choices for the Φ_n, so the sequence $(\Phi_n)_{n=1}^{\infty}$ has as an attractor a non-autonomous self-similar set $K_{\eta(\gamma)}$ and corresponding metric tree $\Omega(\eta(\gamma))$, as defined in §3. This non-autonomous IFS has uniformly bounded contractions and satisfies the OSC with respect to the open interval $(0, 1)$. For notational simplicity, we denote the cylinder sets which compose this metric tree as

$$\mathcal{F}_{\eta(\gamma),n} = \{[j_1, \ldots, j_n] : (j_1, \ldots, j_n) \in \Phi_1 \times \cdots \times \Phi_n\} \quad \text{and} \quad \mathcal{F}_{\eta(\gamma)} = \bigcup_{n=0}^{\infty} \mathcal{F}_{\eta(\gamma),n}.$$

We call $K_{\eta(\gamma)}$ the *symbolic slice* associated with the word γ. If the projected IFS $\{S_{\underline{i},1}\}_{\underline{i} \in \eta(\mathcal{I})}$ satisfies the SSC, then if $x = \eta(\pi(\gamma))$,

$$\{x\} \times K_{\eta(\gamma)} = \eta^{-1}(x) \cap K$$

is precisely the vertical slice of K containing x. In general, $K_{\eta(\gamma)}$ is always contained inside a vertical slice of K. The symbolic fibre $K_{\eta(\gamma)}$ (and its associated Assouad dimension) was introduced and studied in [FR22+, §1.2] in the more general setting of overlapping diagonal carpets.

4.3. Tangents of Gatzouras–Lalley carpets. It turns out that the pointwise Assouad dimension at $x = \pi(\gamma)$ is closely related to the Assouad dimension of the symbolic fibre $K_{\eta(\gamma)}$. In this section, we make this notion precise, and moreover use it to construct large tangents for Gatzouras–Lalley carpets.

In our main result in this section, we prove that approximate squares containing a fixed word $\gamma \in \Omega$ converge in Hausdorff distance to product sets of weak tangents of $K_{\eta(\gamma)}$ with the projection $\eta(K)$, up to some finite distortion and contributions from adjacent approximate squares. First, we define

$$\Phi_{k,\gamma}(x, y) = \big(S^{-1}_{\gamma\upharpoonright_{L_k(\gamma)},1}(x), S^{-1}_{\gamma\upharpoonright_k,2}(y)\big).$$

By choice of $L_k(\gamma)$, the maps $\Phi_{k,\gamma}$ are (up to some constant-size distortion) homotheties. One can think of $\Phi_{k,\gamma}$ as mapping the approximate square $\pi(Q_k(\gamma))$ to the unit square $[0, 1]^2$.

Proposition 4.6. *Let K be a Gatzouras–Lalley carpet and let $\gamma \in \Omega$ be arbitrary. Suppose $(\mathtt{i}_n)_{n=1}^{\infty}$ is any sequence such that $\eta(\mathtt{i}_n) = \eta(\gamma\upharpoonright_n)$. Then*

$$p_{\mathcal{H}}\left(\eta(K) \times (S^{-1}_{\mathtt{i}_n,2}(K_{\eta(\gamma)}) \cap [0, 1]); \Phi_{n,\gamma}(K) \cap [0, 1]^2\right) \lesssim \kappa^n \tag{4.3}$$

where $\kappa = \max\{\frac{\beta_{i,2}}{\beta_{i,1}} : i \in \mathcal{I}\} \in (0, 1)$. Moreover, suppose γ is regular. Then for any $\gamma \in \Omega$ and $F \in \operatorname{Tan}(K, \pi(\gamma))$, there is an $E \in \operatorname{Tan}(K_{\eta(\gamma)})$ and a similarity map h so that $h(F) \subset \eta(K) \times E$.

Proof. We first prove that

$$d_{\mathcal{H}}\left(\eta(K) \times (S^{-1}_{\mathtt{i}_n,2}(K_{\eta(\gamma)}) \cap [0, 1]), \Phi_{n,\gamma}(\pi(Q_n(\gamma)))\right) \lesssim \kappa^n$$

Fix $n \in \mathbb{N}$ and write $k = L_n(\gamma)$. Let $Q_n(\gamma) = P(\gamma\upharpoonright_n, \underline{\mathtt{j}})$ and enumerate $\eta^{-1}(\underline{\mathtt{j}}) = \{\mathtt{j}_1, \ldots, \mathtt{j}_n\} \subset \mathcal{I}^{k-n}$. Observe that $\eta(T_{\mathtt{j}_i}(K)) = S_{\mathtt{j}_i,1}(K)$ does not depend on the choice of $i = 1, \ldots, m$. Now $\Phi_{n,\gamma}(T_{\gamma\upharpoonright_n \mathtt{j}_i}(K))$ is contained in the rectangle $[0, 1] \times S_{\mathtt{j}_i,2}(K)$. Moreover, the rectangle $[0, 1] \times S_{\mathtt{j}_i,2}(K)$ has height $\lesssim \kappa^n$. Therefore

$$d_{\mathcal{H}}\left(\eta(K) \times \bigcup_{i=1}^{m} S_{\mathtt{j}_i,2}([0, 1]), \Phi_{n,\gamma}(Q_n(\gamma))\right) \lesssim \kappa^n. \tag{4.4}$$

But approximating the set $S_{\mathtt{i}_n,2}([0, 1]) \cap K_{\eta(\gamma)}$ at level n with cylinders at level $k = L_n(\gamma)$, using the fact that $\eta(\mathtt{i}_n) = \eta(\gamma\upharpoonright_n)$,

$$d_{\mathcal{H}}\left(S^{-1}_{\mathtt{i}_n,2}(K_{\eta(\gamma)}) \cap [0, 1], \bigcup_{i=1}^{m} S_{\mathtt{j}_i,2}([0, 1])\right) \lesssim \kappa^n. \tag{4.5}$$

Combining (4.4) and (4.5) gives the claim. In particular, noting that $Q_n(\gamma) \subset K$ and $\Phi_{n,\gamma}(Q_n(\gamma)) \subset [0, 1]^2$ gives (4.3).

Now suppose in addition that $x = \pi(\gamma)$ is regular and let $r > 0$ be arbitrary. Since x is regular, there is an $n \in \mathbb{N}$ with $r \leq \beta_{\gamma\upharpoonright_n,1} \lesssim r$ such that

$$B(x, r) \cap K \subset \bigcup_{j=1}^{\ell} T_{\mathtt{i}_j}(K)$$

where

$$\{\mathtt{i}_1, \ldots, \mathtt{i}_\ell\} = \{\mathtt{i} \in \mathcal{I}^n : \eta(\mathtt{i}) = \eta(\gamma|_n) \text{ and } T_{\mathtt{i}}(K) \cap B(x,r) \neq \emptyset\}.$$

Now exactly as before, each rectangle $T_{\mathtt{i}_j}(K)$ has width $\approx r$ and height $\lesssim r\kappa^n$. Therefore identifying $x \in K$ with the analogous point $x \in K_{\eta(\gamma)}$, there is a similarity map h_r with contraction ratio in some interval $[1, c]$ so that

$$p_{\mathcal{H}}\left(r^{-1}(K-x) \cap B(0,1); h_r(\eta(K)) \times r^{-1}(K_{\eta(\gamma)} - x)\right) \lesssim \kappa^n.$$

Now suppose $F \in \operatorname{Tan}(K, x)$ so that $F = \lim_{n\to\infty} r_n^{-1}(K-x) \cap B(0,1)$. Passing to a subsequence, we may assume that the h_{r_n} have contraction ratios converging to some $r_0 \geq 1$. Thus passing again to a subsequence, let $F_0 = \lim_{n\to\infty}(r_0 r_n)^{-1}(K - x) \cap B(0,1)$. Since $r_0 \geq 1$, we have $F \subset F_0$. Passing again to a subsequence, let

$$\lim_{n\to\infty}(r_0 r_n)^{-1}(K_{\eta(\gamma)} - x) \cap B(0,1) = E \in \operatorname{Tan}(K_{\eta(\gamma)}).$$

Thus $r_0^{-1} F \subset F_0 \subset \eta(K) \times E$, as claimed. □

To conclude this section, we establish our general result which guarantees the existence of product-like tangents for arbitrary points in Gatzouras–Lalley carpets.

Proposition 4.7. *Let K be a Gatzouras–Lalley carpet. Then for each $x \in K$, there is an $F \in \operatorname{Tan}(K, x)$ so that*

$$\mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}(F) \gtrsim 1,$$

where $\gamma \in \Omega$ is such that $\pi(\gamma) = x$. In particular,

$$\dim_{\mathrm{A}}(K, x) \geq \max\{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}, \dim_{\mathrm{B}} K\}.$$

Proof. We will construct the set F essentially as a product $\eta(K) \times E$ where E is a *weak* tangent of $K_{\eta(\gamma)}$. First, recall from Corollary 3.3 that $\dim_{\mathrm{A}} K_{\eta(\gamma)} = \dim_{\mathrm{A}} \Omega(\eta(\gamma))$. Thus there is a sequence $(n_k)_{k=1}^\infty$ diverging to infinity and words $\mathtt{i}_k \in \mathcal{I}^{n_k}$ with $\eta(\mathtt{i}_k) = \gamma|_{n_k}$ such that

$$E := \lim_{k\to\infty} S_{\mathtt{i}_k,2}^{-1}(K_{\eta(\gamma)}) \cap [0,1]$$

has $\mathcal{H}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}}(E) \gtrsim 1$.

Thus by Proposition 4.6 applied along the sequence $(\mathtt{i}_k)_{k=1}^\infty$, since the images $\Phi_{n,\gamma}^{-1}([0,1]^2)$ are rectangles with bounded eccentricity containing $\pi(\gamma)$, there is a tangent $F \in \operatorname{Tan}(K, x)$ containing an image of $\eta(K) \times E$ under a bi-Lipschitz map with constants depending only on K. But $\eta(K)$ is Ahlfors–David regular so that

$$\mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}(F) \geq \mathcal{H}^{\dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}}\left(\eta(K) \times E\right) \gtrsim 1$$

as claimed. The result concerning $\dim_{\mathrm{A}}(K, x)$ then follows by Proposition 2.2 and Proposition 2.13. □

4.4. Upper bounds for the pointwise Assouad dimension. We now prove our main upper bound for the pointwise Assouad dimension of Gatzouras–Lalley carpets. As a result of the local inhomogeneity of Gatzouras–Lalley carpets, obtaining good upper bounds requires some care. We will prove a sequence of lemmas which, morally, provide optimal covers for a variety of symbolic objects: these covers will then be combined to obtain our general upper bound for the pointwise Assouad dimension.

We first show that, as a result of the vertical alignment of their component cylinders, pseudo cylinders can essentially be covered by their projection. Recall that $\mathcal{S}$ denotes the set of all approximate squares. Then if $P(\mathtt{i},\underline{\mathtt{j}})$ is any wide pseudo-cylinder, we can write it as a union of the approximate squares in the family

$$\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) = \{Q \in \mathcal{S} : \ Q = P(\mathtt{i},\underline{\mathtt{k}}) \text{ for some } \underline{\mathtt{k}} \in \eta(\mathcal{I}^*) \text{ and } Q \subset P(\mathtt{i},\underline{\mathtt{j}})\}.$$

Since each $Q = P(\mathtt{i},\underline{\mathtt{k}})$ for some $\underline{\mathtt{k}}$, we have $Q \in S(\beta_{\mathtt{i},2})$ so that this family of approximate squares forms a section.

Lemma 4.8. *Let $P(\mathtt{i},\underline{\mathtt{j}})$ be a wide pseudo-cylinder. Then*

$$\#\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) \approx \left(\frac{\beta_{\mathtt{i}\underline{\mathtt{j}},1}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}.$$

Proof. First, enumerate $\mathcal{Q}(\mathtt{i},\underline{\mathtt{j}}) = \{Q_1,\dots,Q_m\}$, and for each $i = 1,\dots,m$, there is a unique $\underline{\mathtt{k}}_i$ so that $Q_i = P(\mathtt{i},\underline{\mathtt{k}}_i)$. Moreover, $\{\underline{\mathtt{k}}_1,\dots,\underline{\mathtt{k}}_m\}$ forms a section relative to $[\underline{\mathtt{j}}]$, so that writing $s = \dim_{\mathrm{B}} \eta(K)$ and recalling that $\eta(K)$ is the attractor of a self-similar IFS satisfying the open set condition,

$$\sum_{i=1}^{m} \beta_{\underline{\mathtt{k}}_i,1}^s = \beta_{\underline{\mathtt{j}},1}^s. \tag{4.6}$$

But $\beta_{\mathtt{i}\underline{\mathtt{k}}_i,1} \approx \beta_{\mathtt{i},2}$ since each Q_i is an approximate square, which gives the desired result. ☐

In the next result, we provide good covers for cylinder sets using approximate squares with diameter bounded above by the height of the corresponding rectangle. Heuristically, a cylinder set can first be decomposed into approximate squares using Lemma 4.8, and an "average" approximate square itself has box dimension the same as the box dimension of K. To make this notion precise, we simply reverse the order: we begin with a good cover for the box dimension of K, and take the image under some word $\mathtt{i}$. The image of each approximate square is a wide pseudo-cylinder, so we may apply Lemma 4.8 to complete the bound.

Lemma 4.9. *Suppose $\mathtt{i} \in \mathcal{I}^*$ and $0 < r \leq \beta_{\mathtt{i},2}$. Then*

$$\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{i}]\} \approx \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\dim_{\mathrm{B}} K} \cdot \left(\frac{\beta_{\mathtt{i},1}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}$$

Proof. Fix $\mathtt{i} \in \mathcal{I}^*$ and $0 < r \leq \beta_{\mathtt{i},2}$. Write $\delta = r/\beta_{\mathtt{i},2}$, so by inspecting the proofs of [LG92, Lemmas 2.1, 2.2, & 2.3], we see that

$$\#\mathcal{S}(\delta) \approx (1/\delta)^{\dim_{\mathrm{B}} K}.$$

Enumerate $\mathcal{S}(\delta) = \{Q_1, \ldots, Q_m\}$ and for each $i = 1, \ldots, m$, we may write $Q_i = P(\mathtt{j}_i, \underline{\mathtt{k}}_i)$ for some $\mathtt{j}_i \in \mathcal{I}^*$ and $\underline{\mathtt{k}}_i \in \eta(\mathcal{I}^*)$. Then for each $i = 1, \ldots, m$,

$$\mathcal{Q}(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i) \subset \mathcal{S}(r) \qquad \text{and} \qquad [\mathtt{i}] = \bigcup_{i=1}^{m} \bigcup_{Q \in \mathcal{Q}(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i)} Q.$$

Thus by Lemma 4.8 applied to each pseudo-cylinder $P(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i)$, since Q_i is an approximate square and $\beta_{\mathtt{j}_i\underline{\mathtt{k}}_i,1} \approx \beta_{\mathtt{j}_i,2}$,

$$\begin{aligned}
\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{i}]\} &= \sum_{i=1}^{m} \#\mathcal{Q}(\mathtt{i}\mathtt{j}_i, \underline{\mathtt{k}}_i) \\
&\approx \sum_{i=1}^{m} \left(\frac{\beta_{\mathtt{i}\mathtt{j}_i\underline{\mathtt{k}}_i,1}}{\beta_{\mathtt{i}\mathtt{j}_i,2}}\right)^{\dim_{\mathrm{B}} \eta(K)} \\
&\approx \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\dim_{\mathrm{B}} K} \cdot \left(\frac{\beta_{\mathtt{i},1}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}
\end{aligned}$$

as claimed. □

To conclude our collection of preliminary lemmas, we use the Assouad dimension of the symbolic fibre $K_{\eta(\gamma)}$ to control the size of "column sections" of approximate squares. We note that the word $\mathtt{i}$ appears in the hypothesis but not the conclusion: this is simply to clarify the application of this lemma when it is used in Proposition 4.11.

Lemma 4.10. *Let $\epsilon > 0$ and $\gamma \in \Omega$ be arbitrary. Suppose $k \in \mathbb{N}$ and $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$. Let $\mathcal{A}$ be any section of $\mathcal{I}^*$ satisfying $\eta^{-1}(\underline{\mathtt{j}}) \preccurlyeq \mathcal{A}$. Then*

$$\sum_{\mathtt{k} \in \mathcal{A}} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)} + \epsilon} \lesssim_{\epsilon,\gamma} 1.$$

Proof. The assumption on the section $\mathcal{A}$ precisely means that $\{\mathtt{i}\mathtt{k} : \mathtt{k} \in \mathcal{A}\}$ is a section relative to $\mathtt{i}$ in $\mathcal{F}_{\eta(\gamma)}$. Then by Proposition 3.6 applied to the metric space $\Omega(\eta(\gamma))$ (recalling that $\dim_{\mathrm{A}} \Omega(\eta(\gamma)) = \dim_{\mathrm{A}} K_{\eta(\gamma)}$ from Corollary 3.3), since $\mathcal{A}$ is a section,

$$\sum_{\mathtt{k} \in \mathcal{A}} \left(\frac{\beta_{\mathtt{i}\mathtt{k},2}}{\beta_{\mathtt{i},2}}\right)^{\dim_{\mathrm{A}} K_{\eta(\gamma)} + \epsilon} \lesssim_{\epsilon,\gamma} 1.$$

Cancelling the $\beta_{\mathtt{i},2}$ gives the desired result. □

Finally, by combining the various counts that we have established earlier in this section, we are now in position to compute the upper bound for the pointwise Assouad dimension.

Let us begin with an intuitive explanation for this proof. Since x is regular, we will reduce the problem of computing covers of balls to computing covers for approximate squares. Thus suppose we fix an approximate square $P(\mathtt{i}, \underline{\mathtt{j}})$, which is the union of cylinders $\{\mathtt{ik} : \eta(\mathtt{k}) = \underline{\mathtt{j}}\}$. We wish to cover this set with approximate squares in $\mathcal{S}(r)$. There are two cases. First, the rectangle corresponding to the cylinder $\mathtt{ik}$ has height greater than or equal to r, in which case we simply keep this cylinder and obtain a good bound for the cover using Lemma 4.9: this is the family $\mathcal{A}_1$. Otherwise, the rectangle is shorter, and we instead want to cover groups of cylinders simultaneously. Such groups are precisely wide pseudo-cylinders corresponding to elements of $\mathcal{A}_2$ and have height r, which we can then cover using Lemma 4.8. These covers are then combined using Lemma 4.10.

Proposition 4.11. *Let K be a Gatzouras–Lalley carpet and suppose $x = \pi(\gamma) \in K$. Then*

$$\dim_{\mathrm{A}}(K, x) \geq \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}\}$$

with equality if x is regular.

Proof. Recalling the general lower bound proven in Proposition 4.7, we must show that

$$\dim_{\mathrm{A}}(K, x) \leq \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{H}} \eta(K) + \dim_{\mathrm{A}} K_{\eta(\gamma)}\} =: \zeta$$

when x is regular. We obtain this bound by a direct covering argument. We will prove that for any $k \in \mathbb{N}$ and approximate square $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$, if $0 < r \leq \beta_{\mathtt{i},2}$, then

$$\#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma)\} \lesssim \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta}. \tag{4.7}$$

Assuming this, since x is regular, for any ball $B(x, R)$, there is an $R' \lesssim R$ and at most two approximate squares $Q_1, Q_2 \in \mathcal{S}(R')$ lying in the same column such that $B(x, R) \subset \pi(Q_1) \cup \pi(Q_2)$. Since Q_1, Q_2 lie in the same column, $Q_j = Q_{k_j}(\gamma_j)$ for some $k_j \in \mathbb{N}$ where $\eta(\gamma_j) = \eta(\gamma)$. Moreover, if $0 < r \leq R$ and $Q \in \mathcal{S}(r)$ is arbitrary, then $\operatorname{diam} \pi(Q) \lesssim r$. Thus (4.7) immediately gives the correct bound, up to a constant factor, for $N_r(B(x, R) \cap K)$.

It remains to prove (4.7). Fix an approximate square $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$ and suppose $0 < r \leq \beta_{\mathtt{i},2}$ is arbitrary. First, let

$$\mathcal{A}_0 = \eta^{-1}(\underline{\mathtt{j}}) \vee \mathcal{F}_{\eta(\gamma)}(r/\beta_{\mathtt{i},2}) \qquad \text{and} \qquad \mathcal{A} = \{\mathtt{ik} : \mathtt{k} \in \mathcal{A}_0\}.$$

We then decompose $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where

$$\mathcal{A}_1 = \mathcal{A} \setminus \mathcal{F}_{\eta(\gamma)}(r) \qquad \text{and} \qquad \mathcal{A}_2 = \mathcal{A} \cap \mathcal{F}_{\eta(\gamma)}(r).$$

First, suppose $\mathtt{ik} \in \mathcal{A}_1$. Then, by definition, $\beta_{\mathtt{ik},2} > r$ which, by definition of $\mathcal{A}_0$, implies that $\eta(\mathtt{k}) = \underline{\mathtt{j}}$. Thus by Lemma 4.9 applied to the cylinder $\mathtt{ik}$ and scale r, since $\dim_{\mathrm{B}} \eta(K) \leq \dim_{\mathrm{B}} K$ and $\beta_{\mathtt{ik},1} \approx \beta_{\mathtt{i},2}$,

$$\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{ik}]\} \approx \left(\frac{\beta_{\mathtt{ik},2}}{r}\right)^{\dim_{\mathrm{B}} K} \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}. \tag{4.8}$$

Otherwise, suppose $\mathtt{ik} \in \mathcal{A}_2 \subset \mathcal{F}_{\eta(\gamma)}(r)$. Since $\eta^{-1}(\underline{\mathtt{j}}) \preccurlyeq \mathcal{A}_0$, there is a $\underline{\mathtt{j}}'$ so that $\eta(\mathtt{k})\underline{\mathtt{j}}' = \underline{\mathtt{j}}$. Thus choice of $\underline{\mathtt{j}}'$ ensures that

$$P(\mathtt{ik}, \underline{\mathtt{j}}') = Q_k(\gamma) \cap [\mathtt{ik}].$$

Thus by Lemma 4.8 and since $Q_k(\gamma) = P(\mathtt{i}, \underline{\mathtt{j}})$ is an approximate square,

$$\#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma) \cap [\mathtt{ik}]\} = \#\mathcal{Q}(\mathtt{ik}, \underline{\mathtt{j}}' \approx \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)}. \tag{4.9}$$

Thus by applying (4.8) and (4.9) to the respective components and recalling that $\beta_{\mathtt{ik},2} \approx r$ whenever $\mathtt{ik} \in \mathcal{A}_2$,

$$\begin{aligned}
&\#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma)\} \\
&\qquad = \sum_{\mathtt{ik} \in \mathcal{A}_1} \#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{ik}]\} + \sum_{\mathtt{ik} \in \mathcal{A}_2} \#\{Q \in \mathcal{S}(r) : Q \subset Q_k(\gamma) \cap [\mathtt{ik}]\} \\
&\qquad \approx \sum_{\mathtt{ik} \in \mathcal{A}_1} \left(\frac{\beta_{\mathtt{ik},2}}{r}\right)^{\dim_{\mathrm{B}} K} \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)} + \sum_{\mathtt{ik} \in \mathcal{A}_2} \left(\frac{1}{\beta_{\mathtt{k},2}}\right)^{\dim_{\mathrm{B}} \eta(K)} \\
&\qquad \lesssim \sum_{\mathtt{ik} \in \mathcal{A}_1} \left(\frac{\beta_{\mathtt{ik},2}}{r}\right)^{\zeta} \left(\frac{\beta_{\mathtt{i},2}}{\beta_{\mathtt{ik},2}}\right)^{\zeta} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}} + \sum_{\mathtt{ik} \in \mathcal{A}_2} \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}} \\
&\qquad = \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta} \sum_{\mathtt{k} \in \mathcal{A}_0} \beta_{\mathtt{k},2}^{\dim_{\mathrm{A}} K_{\eta(\gamma)}} \\
&\qquad \lesssim \left(\frac{\beta_{\mathtt{i},2}}{r}\right)^{\zeta}
\end{aligned}$$

where in the last line follows by Lemma 4.10 applied to the section $\mathcal{A}_0$. Thus (4.7) follows, and therefore our desired result. □

4.5. Dimensions of level sets of pointwise Assouad dimension. Given an index $i \in \mathcal{I}$, let $\Phi_{\eta(i)}$ denote the IFS corresponding to the column containing the index i, that is

$$\Phi_{\eta(i)} = \{S_{j,2} : j \in \mathcal{I} \text{ and } \eta(j) = \eta(i)\}.$$

Now given a word $\gamma = (i_n)_{n=1}^{\infty} \in \Omega$, recall that the symbolic slice $K_{\eta(\gamma)}$ is the non-autonomous self-similar set associated with the IFS $\{\Phi_{\eta(i_n)}\}_{n=1}^{\infty}$. Since there are only finitely many choices for the $\Phi_{\eta(i_n)}$, the hypotheses of Theorem 3.7 are automatically satisfied and

$$\dim_{\mathrm{A}} K_{\eta(\gamma)} = \lim_{m \to \infty} \sup_{n \in \mathbb{N}} \theta_{\eta(\gamma)}(n, m)$$

where

$$\sum_{(j_1,\ldots,j_m)\in\eta^{-1}(\eta(i_1,\ldots,i_n))}\prod_{k=1}^{m}\beta_{j_k,2}^{\theta_{\eta(\gamma)}(n,m)}=1.$$

We now obtain our main formula for the pointwise Assouad dimension of arbitrary points in Gatzouras–Lalley carpets.

Theorem 4.12. *Let K be a Gatzouras–Lalley carpet. Then for every $x\in K$ with $x=\pi(\gamma)$, there is an $F\in\operatorname{Tan}(K,x)$ with $\mathcal{H}^s(F)\gtrsim 1$ where*

$$\begin{aligned}s &:= \dim_{\mathrm{B}}\eta(K)+\dim_{\mathrm{A}}K_{\eta(\gamma)}\\ &= \dim_{\mathrm{B}}\eta(K)+\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\theta_{\eta(\gamma)}(n,m)\end{aligned}$$

In particular,

$$\max\{\dim_{\mathrm{H}}F:F\in\operatorname{Tan}(K,x)\}\geq s\quad\textit{and}\quad\dim_{\mathrm{A}}(K,x)\geq\max\{s,\dim_{\mathrm{B}}\eta(K)\}$$

where both inequalities are equalities if x is regular. In particular, if $\eta(K)$ satisfies the strong separation condition then equality holds for all $x\in K$.

Proof. By Proposition 4.7, there is an $F\in\operatorname{Tan}(K,x)$ so that

$$\mathcal{H}^{\dim_{\mathrm{H}}\eta(K)+\dim_{\mathrm{A}}K_{\eta(\gamma)}}(F)\gtrsim 1.$$

Moreover, $\dim_{\mathrm{A}}K_{\eta(\gamma)}=\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\theta_{\eta(\gamma)}(n,m)$ by Theorem 3.7. The formula for $\dim_{\mathrm{A}}(K,x)$, including the case when x is regular, then follows by Proposition 4.11.

If x is regular, it moreover follows from Proposition 4.6 that for any $F\in\operatorname{Tan}(K,x)$, there is a similarity map h and a weak tangent $E\in\operatorname{Tan}(K_{\eta(\gamma)})$ so that $h(F)\subset\eta(K)\times E$. Since $\dim_{\mathrm{B}}\eta(K)=\dim_{\mathrm{H}}\eta(K)$,

$$\dim_{\mathrm{H}}F=\dim_{\mathrm{H}}h(F)\leq\dim_{\mathrm{B}}\eta(K)+\dim_{\mathrm{H}}E\leq\dim_{\mathrm{B}}\eta(K)+\dim_{\mathrm{A}}K_{\eta(\gamma)}$$

as required.

Finally, we recall that if $\eta(K)$ satisfies the strong separation condition, then each $x\in K$ is regular by Lemma 4.4 (i). □

Our next goal is to prove that the set of pointwise Assouad dimensions forms an interval. The main observation required in the proof is a stability result for the expression $\theta_{\eta(\gamma)}(n,m)$ when m is large. In order to facilitate the proof, we establish some notation. First suppose $m\in\mathbb{N}$ and $\mathtt{i}\in\mathcal{I}^m$. Define $\phi_{\mathtt{i}}\colon[0,1]\to\mathbb{R}$ by

$$\phi_{\mathtt{i}}(s)=\sum_{\substack{\mathtt{j}\in\mathcal{I}^n\\ \eta(\mathtt{j})=\eta(\mathtt{i})}}\beta_{\mathtt{j},2}^s.$$

Since $\phi_{\mathtt{i}}$ is strictly decreasing with $\phi_{\mathtt{i}}(0)>1$ and $\phi_{\mathtt{i}}(1)<1$, there is a unique $t(\mathtt{i})$ so that $\phi_{\mathtt{i}}(t(\mathtt{i}))=1$. Note that $0<s_{\min}\leq t(\mathtt{i})\leq s_{\max}<1$ where $s_{\min}=\min\{t(i):$

$i \in \mathcal{I}\}$ and $s_{\max} = \max\{t(i) : i \in \mathcal{I}\}$. Of course, the function t is chosen precisely so that

$$\theta_{\eta(\gamma)}(n, m) = t(\gamma_{n+1}, \ldots, \gamma_{n+m}).$$

We can now prove, essentially, that the set of fibre dimensions form an interval.

Lemma 4.13. *Let K be a Gatzouras–Lalley carpet and suppose $\dim_{\mathrm{L}} K < \alpha < \dim_{\mathrm{A}} K$. Then for all $k_0 \in \mathbb{N}$ sufficiently large, for all $n \in \mathbb{N}$ there is $\mathtt{i}_n \in \mathcal{B}_{k_0}^n \subset \mathcal{I}^{k_0 n}$ satisfying*

$$\lim_{m \to \infty} \sup_{n \in \mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) = \alpha - \dim_{\mathrm{B}} \eta(K).$$

Proof. Let $\mathtt{i} \in \mathcal{I}^m$ and $j \in \mathcal{I}$ be arbitrary. We first show that $|t(\mathtt{i}j) - t(\mathtt{i})|$ converges to zero uniformly as m diverges to infinity. First, if $j \in \mathcal{I}$ is arbitrary, then

$$\phi_{\mathtt{i}j}(t(\mathtt{i})) = \sum_{\substack{i \in \mathcal{I} \\ \eta(i) = \eta(j)}} \beta_{i,2}^{t(\mathtt{i})} \approx 1. \tag{4.10}$$

On the other hand,

$$\phi_{\mathtt{i}j}(t(\mathtt{i}) + \epsilon) \le \phi_{\mathtt{i}j}(t(\mathtt{i})) \cdot (\min\{\beta_{i,2} : i \in \mathcal{I}\})^{m\epsilon}$$

so that, if $t(\mathtt{i}) + \epsilon \ge t(\mathtt{i}j)$, applying (4.10), we observe that $(\min\{\beta_{i,2} : i \in \mathcal{I}\})^{m\epsilon} \approx 1$ which forces $\epsilon \approx 1/m$. The same argument also holds for the lower bound. Iterating the above bound, we have therefore proven that for any $m, k \in \mathbb{N}$, $\mathtt{i} \in \mathcal{I}^m$, and $\mathtt{j} \in \mathcal{I}^k$,

$$|t(\mathtt{i}\mathtt{j}) - t(\mathtt{i})| \lesssim \frac{k}{m}. \tag{4.11}$$

We now proceed with our general construction. First, fixing any interior word $\mathtt{j} \in \mathcal{I}^*$ and $i \in \mathcal{I}$ so that $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + t(i)$,

$$\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + \lim_{k \to \infty} t(\mathtt{j} i^k);$$

and similarly for the lower dimension. Thus for all sufficiently large k_0, there are words $\mathtt{j}_L, \mathtt{j}_A \in \mathcal{B}_{k_0}$ so that

$$\dim_{\mathrm{B}} \eta(K) + t(\mathtt{j}_L) < \alpha < \dim_{\mathrm{B}} \eta(K) + t(\mathtt{j}_A).$$

We inductively construct $(\mathtt{j}_{L,k}, \mathtt{j}_{A,k})_{k=1}^\infty$ so that, for each $k \in \mathbb{N}$,

1. $\alpha - \dim_{\mathrm{B}} \eta(K) - \frac{1}{k} \le t(\mathtt{j}_{L,k}) \le \alpha - \dim_{\mathrm{B}} \eta(K)$,
2. $\alpha - \dim_{\mathrm{B}} \eta(K) \le t(\mathtt{j}_{A,k}) \le \dim_{\mathrm{A}} K + \dim_{\mathrm{B}} \eta(K) + \frac{1}{k}$,
3. $\mathtt{j}_{L,k}, \mathtt{j}_{A,k} \in \mathcal{B}_{k_0}^*$ and, for $k \ge 2$, $\mathtt{j}_{L,k}, \mathtt{j}_{A,k} \in \{\mathtt{j}_{L,k-1}, \mathtt{j}_{A,k-1}\}^*$, and
4. $|\mathtt{j}_{L,k}| \ge k$ and $|\mathtt{j}_{A,k}| \ge k$.

First, set $\mathtt{j}_{L,1} = \mathtt{j}_L$ and $\mathtt{j}_{A,1} = \mathtt{j}_A$ which clearly satisfy the desired properties. Now suppose we have constructed $\mathtt{j}_{L,k}$ and $\mathtt{j}_{A,k}$. Since $t(\mathtt{j}_{A,k}) \geq \alpha - \dim_{\mathrm{B}} \eta(K)$, for any $m \in \mathbb{N}$,

$$\lim_{n\to\infty} t(\mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^n) \geq \dim_{\mathrm{B}} \eta(K) - \alpha.$$

Moreover, $t(\mathtt{j}_{L,k}^m) \leq \dim_{\mathrm{B}} \eta(K) - \alpha$ and, by taking $m \geq k$ sufficiently large and applying (4.11), for all $n \in \mathbb{N}$ sufficiently large,

$$|t(\mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^{n+1}) - t(\mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^n)| \leq \frac{1}{k+2} < \frac{1}{k+1}.$$

Combining these two observations, there is a pair m, n so that $\mathtt{j}_{A,k+1} \coloneqq \mathtt{j}_{L,k}^m \mathtt{j}_{A,k}^n \in \mathcal{B}_{k_0}^*$ satisfies conditions 1 and 4. The identical argument gives $\mathtt{j}_{L,k+1} \in \mathcal{B}_{k_0}^*$ satisfying 2, as claimed.

To complete the proof, since $\mathtt{j}_{L,k} \in \mathcal{B}_{k_0}^*$ for all $k \in \mathbb{N}$, we may identify the sequence $(\mathtt{j}_{L,k})_{k=1}^\infty$ with a sequence $(\mathtt{i}_n)_{n=1}^\infty$ where $\mathtt{i}_n \in \mathcal{B}_{k_0}$ for all $n \in \mathbb{N}$. It immediately follows from 4 that

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) \geq \alpha - \dim_{\mathrm{B}} \eta(K).$$

To establish the converse bound, it suffices to show for every $k \in \mathbb{N}$ that

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \frac{1}{k}.$$

By 3, for all $k \in \mathbb{N}$, there is a $K \in \mathbb{N}$ so that for all $n \geq K$, $\mathtt{i}_n \in \{\mathtt{j}_{L,k}, \mathtt{j}_{A,k}\}^*$. For each $\ell \in \mathbb{N}$, write $\mathtt{k}_\ell = \mathtt{i}_{K\ell+1} \cdots \mathtt{i}_{K(\ell+1)}$ and note that $\mathtt{k}_\ell \in \{\mathtt{j}_{L,k}, \mathtt{j}_{A,k}\}^*$ for all $\ell \in \mathbb{N}$. Thus for any $n, m \in \mathbb{N}$,

$$t(\mathtt{k}_{\ell+1} \cdots \mathtt{k}_{\ell+m}) \leq \frac{1}{m} \sum_{i=1}^m t(\mathtt{k}_{\ell+i}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \frac{1}{k}.$$

But by Lemma 3.4 and the subadditivity property of t established in Theorem 3.7,

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{i}_{n+1} \cdots \mathtt{i}_{n+m}) = \lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{k}_{n+1} \cdots \mathtt{k}_{n+m})$$

which gives the claim. $\square$

To conclude this section, we assemble the results proven in the prior two sections to obtain our main result.

Theorem 4.14. *Let K be a Gatzouras–Lalley carpet. Then for any $\dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{A}} K$,*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\} = \dim_{\mathrm{H}} K. \tag{4.12}$$

Otherwise, if $\alpha \notin [\dim_{\mathrm{B}} K, \dim_{\mathrm{A}} K]$, then $\{x \in K : \dim_{\mathrm{A}}(K, x) = \alpha\} = \emptyset$. However,

$$\mathcal{H}^{\dim_{\mathrm{H}} K}\big(\{x \in K : \dim_{\mathrm{A}}(K, x) \neq \dim_{\mathrm{A}} K\}\big) = 0. \tag{4.13}$$

Proof. Note that if $\dim_{\mathrm{B}} K = \dim_{\mathrm{A}} K$, then $\dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K$ for all $x \in K$ and the results are clearly true. Thus we may assume that $\dim_{\mathrm{H}} K < \dim_{\mathrm{B}} K < \dim_{\mathrm{A}} K$.

We first establish (4.12). Let $\epsilon > 0$ be arbitrary and $\dim_{\mathrm{B}} K \leq \alpha \leq \dim_{\mathrm{A}} K$. Apply Proposition 4.5 and get $k \in \mathbb{N}$ and a family $\mathcal{J} \subset \mathcal{B}_k$ with corresponding attractor K_ϵ satisfying $\dim_{\mathrm{H}} K - \epsilon \leq \dim_{\mathrm{H}} K_\epsilon = \dim_{\mathrm{A}} K_\epsilon$ and $\dim_{\mathrm{B}} \eta(K) - \epsilon \leq \dim_{\mathrm{B}} \eta(K_\epsilon)$. If $\alpha < \dim_{\mathrm{A}} K$, iterating the system if necessary, by Lemma 4.13 get a sequence $(\mathtt{i}_n)_{n=1}^\infty$ with $\mathtt{i}_n \in \mathcal{B}_k$ for all $n \in \mathbb{N}$ and moreover

$$\lim_{m\to\infty} \sup_{n\in\mathbb{N}} t(\mathtt{i}_{n+1}\cdots\mathtt{i}_{n+m}) = \alpha - \dim_{\mathrm{B}} \eta(K). \tag{4.14}$$

If instead $\alpha = \dim_{\mathrm{A}} K$, instead simply take $\mathtt{i}_n = i_0^k$ where $i_0 \in \mathcal{I}$ is any word such that $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + t(i_0)$. Note that $t(\mathtt{j}) = \dim_{\mathrm{A}} K_\epsilon - \dim_{\mathrm{B}} \eta(K_\epsilon)$ for any $\mathtt{j} \in \mathcal{J}$. Thus by taking ϵ to be sufficiently small, we may assume that $t(\mathtt{j}) \leq \alpha - \dim_{\mathrm{B}} \eta(K)$ for all $\mathtt{j} \in \mathcal{J}$.

Now, let $(N_k)_{k=1}^\infty$ be a sequence of natural numbers satisfying $\lim_{k\to\infty} N_k/k = \infty$ and write

$$\Omega_0 = \prod_{k=1}^{\infty} \mathcal{J}^{N_k} \times \{\mathtt{i}_1\} \times \cdots \times \{\mathtt{i}_k\}.$$

By taking each N_k to be sufficiently large, we may ensure that $\dim_{\mathrm{H}} \pi(\Omega_0) \geq \dim_{\mathrm{H}} K_\epsilon - \epsilon$. Fix $\gamma \in \Omega_0$: it remains to verify that $\dim_{\mathrm{A}}(K, \pi(\gamma)) = \alpha$. Since $\gamma \in \mathcal{B}_k^{\mathbb{N}}$, $\pi(\gamma)$ is a regular point of K by Lemma 4.4 (ii). By passing to the subsystem induced by $\mathcal{B}_k \subset \mathcal{I}^k$, write $\gamma = (\mathtt{k}_k)_{k=1}^\infty$ where $\mathtt{k}_k \in \mathcal{B}_k$. Thus by Theorem 4.12 and Lemma 3.4,

$$\dim_{\mathrm{A}}(K,x) = \max\bigl\{\dim_{\mathrm{B}} K, \lim_{m\to\infty}\sup_{n\in\mathbb{N}} t(\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m})\bigr\}.$$

Since $\mathtt{i}_1\cdots\mathtt{i}_m$ appears as a subword of of γ for arbitrarily large m, by (4.14) and since $\alpha > \dim_{\mathrm{B}} K$, it follows that $\dim_{\mathrm{A}}(K,x) \geq \alpha$.

We now obtain the upper bound. Let $\epsilon > 0$ be arbitrary. By (4.14), there is an $\ell_0 \in \mathbb{N}$ so that whenever $\ell \geq \ell_0$, we have $t(\mathtt{i}_{j+1}\cdots\mathtt{i}_{j+\ell}) \leq \alpha - \dim_{\mathrm{B}} \eta(K) + \epsilon$. Let m be sufficiently large so that $\ell_0/m \leq \epsilon$. Since $\lim_{k\to\infty} N_k/k = \infty$, for all n sufficiently large, there is a $j \in \mathbb{N}$ so that

$$\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m} = \mathtt{j}_1\cdots\mathtt{j}_{m-\ell}\mathtt{i}_{j+1}\cdots\mathtt{i}_{j+\ell}.$$

Thus for m, n sufficiently large, if $\ell \geq \ell_0$,

$$\begin{aligned} t(\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m}) &\leq \frac{(m-\ell)\cdot t(\mathtt{j}_1\cdots\mathtt{j}_{m-\ell}) + \ell\cdot t(\mathtt{i}_{j+1}\cdots\mathtt{i}_{j+\ell})}{m} \\ &\leq \frac{m-\ell}{m}\cdot(\alpha - \dim_{\mathrm{B}}\eta(K)) + \frac{\ell}{m}(\alpha - \dim_{\mathrm{B}}\eta(K) + \epsilon) \\ &\leq \alpha - \dim_{\mathrm{B}}\eta(K) + \epsilon \end{aligned}$$

and similarly if $\ell < \ell_0$, recalling that $t(\mathtt{i}_{j+1}\cdots\mathtt{i}_{j+\ell}) \leq 1$,

$$t(\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m}) \leq \alpha - \dim_{\mathrm{B}}\eta(K) + \frac{\ell_0}{m} \leq \alpha - \dim_{\mathrm{B}}\eta(K) + \epsilon.$$

Therefore

$$\limsup_{m\to\infty}\limsup_{n\to\infty} t(\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m}) \leq \alpha - \dim_{\mathrm{B}}\eta(K)+\epsilon$$

and since $\epsilon > 0$ was arbitrary,

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}} t(\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m}) = \limsup_{m\to\infty}\limsup_{n\to\infty} t(\mathtt{k}_{n+1}\cdots\mathtt{k}_{n+m}) \leq \alpha - \dim_{\mathrm{B}}\eta(K)$$

so that $\dim_{\mathrm{A}}(K,x) \leq \alpha$, as claimed. Of course, we recall as well that $\dim_{\mathrm{B}} K \leq \dim_{\mathrm{A}}(K,x) \leq \dim_{\mathrm{A}} K$ by Proposition 2.13.

We finally consider the points x such that $\dim_{\mathrm{A}}(K,x) < \dim_{\mathrm{A}} K$. Let $i_0 \in \mathcal{I}$ be such that $\dim_{\mathrm{A}} K = \dim_{\mathrm{B}}\eta(K) + t(i_0)$. Let

$$\mathcal{J}_M \coloneqq \{(i_1,\ldots,i_M)\in\mathcal{I}^M : (i_1,\ldots,i_M)\neq(i_0,\ldots,i_0)\}$$

have attractor $K_M \subset K$. Since $\mathcal{J}_M$ is a proper subsystem, $\dim_{\mathrm{H}} K_M < \dim_{\mathrm{H}} K$ so that $\mathcal{H}^{\dim_{\mathrm{H}} K}(K_M) = 0$. Now let $x\in K$ have $\dim_{\mathrm{A}}(K,x) < \dim_{\mathrm{A}} K$. Suppose $x = \pi(\gamma)$ where $\gamma = (i_n)_{n=1}^\infty$, so that

$$\dim_{\mathrm{A}}(K,x) \geq \max\left\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}}\eta(K) + \lim_{m\to\infty}\sup_{n\in\mathbb{N}} t(i_{n+1},\ldots,i_{n+m})\right\}.$$

Since $\dim_{\mathrm{A}}(K,x) < \dim_{\mathrm{A}} K$,

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}} t(i_{n+1},\ldots,i_{n+m}) < t(i_0).$$

In particular, there is a constant M so that γ does not contain i_0^M as a subword. Thus $x\in K_M$ for some M and therefore

$$\mathcal{H}^{\dim_{\mathrm{H}} K}\left(\{x\in K : \dim_{\mathrm{A}}(K,x) < \dim_{\mathrm{A}} K\}\right) \leq \sum_{M=1}^{\infty}\mathcal{H}^{\dim_{\mathrm{H}} K}(K_M) = 0$$

as required. □

Remark 4.15. We recall that if K is a Gatzouras–Lalley carpet, then $\mathcal{H}^{\dim_{\mathrm{H}} K}(K) > 0$, with $\mathcal{H}^{\dim_{\mathrm{H}} K}(K) < \infty$ if and only if K is Ahlfors regular; see [LG92]. In particular, the positivity of the Hausdorff measure guarantees that the claim (4.13) in Theorem 4.14 is not vacuous; and, if the Hausdorff measure is finite, Theorem 4.14 is trivial.

5. TANGENT STRUCTURE AND DIMENSION OF BARAŃSKI CARPETS

5.1. Dimensions and decompositions of Barański carpets. Recall the definition of the Barański carpet and basic notation from §4.1. Suppose K is a Barański carpet and $\gamma\in\Omega$ is arbitrary. For each $k\in\mathbb{N}$, we define a probability vector $\boldsymbol{\xi}_k(\gamma)$ by the rule

$$\boldsymbol{\xi}_k(\gamma)_i = \frac{\#\{1\leq \ell\leq k : \gamma_\ell = i\}}{k} \quad \text{for each } i\in\mathcal{I}.$$

In other words, $\boldsymbol{\xi}_k(\gamma)$ is the distribution of the letter frequencies in the first k letters of γ. We then define

$$\Gamma_k(\gamma) = \frac{\chi_1(\boldsymbol{\xi}_k(\gamma))}{\chi_2(\boldsymbol{\xi}_k(\gamma))}.$$

The function Γ_k induces a partition $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ by

$$\begin{aligned}
\Omega_0 &= \{\gamma : \liminf_{k\to\infty} \Gamma_k(\gamma) \leq 1 \leq \limsup_{k\to\infty} \Gamma_k(\gamma)\} \\
\Omega_1 &= \{\gamma : \limsup_{k\to\infty} \Gamma_k(\gamma) < 1\} \\
\Omega_2 &= \{\gamma : 1 < \liminf_{k\to\infty} \Gamma_k(\gamma)\}.
\end{aligned}$$

We now recall the dimensional formula for a general Barański carpet. First, we decompose $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$ where

$$\mathcal{P}_j = \{\boldsymbol{w} \in \mathcal{P} : \chi_j(\boldsymbol{w}) \leq \chi_{j'}(\boldsymbol{w})\}.$$

Now given a measure $\boldsymbol{w} \in \mathcal{P}_j$, recall [Bar07, Corollary 5.2] which states that

$$\dim_{\mathrm{H}} \pi_* \boldsymbol{w}^{\mathbb{N}} = \frac{H(\eta_j(\boldsymbol{w}))}{\chi_j(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_j(\boldsymbol{w}))}{\chi_{j'}(\boldsymbol{w})}.$$

Here and for the remainder of this document, for notational simplicity, given $j = 1$ we write $j' = 2$ and given $j = 2$ we write $j' = 1$.

We also introduce some notation for symbolic slices both in the horizontal and vertical directions. Given $\gamma \in \Omega$ and $j \in 1, 2$, let $\theta_{\eta_j(\gamma),j}$ be defined by the rule

$$\sum_{(j_1,\ldots,j_m)\in\eta_j^{-1}(\eta_j(i_1,\ldots,i_n))} \prod_{k=1}^m \beta_{j_k,j}^{\theta_{\eta_j(\gamma),j}(n,m)} = 1.$$

The value $\theta_{\eta(\gamma)} = \theta_{\eta_1(\gamma),1}$ was defined previously in the context of a Gatzouras–Lalley carpet. As is the case with a Gatzouras–Lalley carpet, if we denote by $K_{\eta_j(\gamma),j}$ the non-autonomous self-similar set associated with the non-autonomous self-similar IFS $\{S_{i,j} : i \in \eta^{-1}(\eta(\gamma_k))\}_{k=1}^\infty$, then

$$\dim_{\mathrm{A}} K_{\eta_j(\gamma),j} = \lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta_j(\gamma),j}(n,m).$$

Assuming $\eta_1(K)$ (resp. $\eta_2(K)$) satisfies the SSC, then $K_{\eta_1(\gamma),1}$ (resp. $K_{\eta_2(\gamma),2}$) is precisely the intersection of K with the vertical (resp. horizontal) line containing $x = \pi(\gamma)$. We now recall [Fra14, Theorem 2.12] concerning the Assouad dimension and the main result of [Bar07] on the Hausdorff dimensions of Barański carpets. While this result is not stated explicitly, the relevant details can be obtained directly by inspecting the proof.

Proposition 5.1 ([Bar07; Fra14]). *Let K be a Barański carpet such that $\Omega_1 \neq \emptyset$ and $\Omega_2 \neq \emptyset$. Then:*

(i) *For each* $j = 1, 2$,

$$\dim_{\mathrm{H}} \pi(\Omega_0 \cup \Omega_j) \leq d_j$$

where

$$d_j = \max_{\boldsymbol{w} \in \mathcal{P}_j} \left(\frac{H(\eta_j(\boldsymbol{w}))}{\chi_j(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_j(\boldsymbol{w}))}{\chi_{j'}(\boldsymbol{w})} \right).$$

In particular, $\dim_{\mathrm{H}} K = \max\{d_1, d_2\}$.

(ii) *We have*

$$\dim_{\mathrm{A}} K = \dim_{\mathrm{B}} \eta(K) + \max_{j=1,2} \{t_j\}$$

where

$$t_j = \max_{\underline{\ell} \in \eta_j(\mathcal{I})} t_j(\underline{\ell})$$

and $t_j(\underline{\ell})$ *is the unique solution to the equation*

$$\sum_{j \in \eta_j^{-1}(\underline{\ell})} \beta_{j,2}^{t_j(\ell)} = 1.$$

5.2. Pointwise Assouad dimension along uniformly contracting sequences. In this section, we state a generalization of our results on Gatzouras–Lalley carpets to Barański carpets, with the caveat that we restrict our attention to points coded by sequences which contract uniformly in one direction. The arguments are similar to the Gatzouras–Lalley case so we only include detail when the proofs diverge. Handling more general sequences would result in a more complicated formula for the pointwise Assouad dimension depending on the scales at which the contraction ratio is greater in one direction than the other, which we will not treat here.

We begin by defining the analogues of pseudo-cylinders and approximate squares. Fix $j = 1, 2$. Suppose $\mathtt{i} \in \mathcal{I}^k$ and $\underline{\mathtt{j}} \in \eta_j(\mathcal{I}^\ell)$. We then write

$$P_j(\mathtt{i}, \underline{\mathtt{j}}) = \{\gamma = (i_n)_{n=1}^\infty \in \Omega : (i_1, \ldots, i_k) = \mathtt{i} \text{ and } \eta(i_{k+1}, \ldots, i_{k+l}) = \underline{\mathtt{j}}\}.$$

We call a pseudo-cylinder *wide* if $\beta_{\mathtt{i}\underline{\mathtt{j}},j} \geq \beta_{\mathtt{i},j'}$; otherwise, we call the pseudo-cylinder *tall*.

Now let $\gamma \in \Omega$ be arbitrary and let $k \in \mathbb{N}$. Let j be chosen so that $\beta_{\gamma\restriction_k,j} \geq \beta_{\gamma\restriction_k,j'}$. We then let $L_k(\gamma) \geq k$ denote the maximal integer so that

$$\beta_{\gamma\restriction_{L_{k,j}(\gamma)},j} \geq \beta_{\gamma\restriction_k,j'}.$$

Write $\gamma\restriction_{L_{k,j}(\gamma)} = \mathtt{i}\mathtt{j}$ and define the approximate square

$$Q_k(\gamma) = P_j(\mathtt{i}, \eta_j(\mathtt{j})).$$

Similarly to the Gatzouras–Lalley case, the collection of approximate squares forms a metric tree when equipped with the valuation $\rho(P_j(\mathtt{i}, \eta_j(\mathtt{j}))) = \beta_{\mathtt{i},j'}$. Note that for each approximate square Q, there is a unique choice for j except precisely when $\beta_{\gamma|_k,j} = \beta_{\gamma|_k,j'}$, so indeed ρ is well-defined.

Similarly as in the Gatzouras–Lalley case, given a pseudo-cylinder $P_j(\mathtt{i}, \underline{\mathtt{j}})$, we write

$$\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) = \max\{\mathcal{A} : \mathcal{A} \text{ is a section of } \mathcal{S} \text{ relative to } P_j(\mathtt{i}, \underline{\mathtt{j}})\}$$

where $\mathcal{S}$ is the collection of all approximate squares and the maximum is with respect to the partial ordering on sections. That the maximum always exists follows from the properties of the join. In the case when the pseudo-cylinder is wide, this coincides precisely with the definition in the Gatzouras–Lalley case.

However, unlike in the Gatzouras–Lalley case, we will also have to handle tall pseudo-cylinders, which have a more complex structure. This additional structure is handled in the following covering lemma.

Lemma 5.2. *(i) Let $P_j(\mathtt{i}, \underline{\mathtt{j}})$ be a wide pseudo-cylinder. Then*

$$\#\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) \approx \left(\frac{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}{\beta_{\mathtt{i},j'}}\right)^{\dim_{\mathrm{B}} \eta_j(K)}.$$

(ii) Let $P_j(\mathtt{i}, \underline{\mathtt{j}})$ be a tall pseudo-cylinder. Then

$$\#\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) \lesssim \left(\frac{\beta_{\mathtt{i},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)}.$$

(iii) Let $\epsilon > 0$ be arbitrary. Suppose $\mathtt{i} \in \mathcal{I}^*$ and let j be chosen so that $\beta_{\mathtt{i},j'} \leq \beta_{\mathtt{i},j}$. Let $0 < r \leq \beta_{\mathtt{i},j}$. Then

$$\#\{Q \in \mathcal{S}(r) : Q \subset [\mathtt{i}]\} \lesssim_{\epsilon} \left(\frac{\beta_{\mathtt{i},j'}}{r}\right)^{\dim_{\mathrm{B}} K+\epsilon} \cdot \left(\frac{\beta_{\mathtt{i},j}}{\beta_{\mathtt{i},j'}}\right)^{\dim_{\mathrm{B}} \eta_j(K)}.$$

(iv) Let $\epsilon > 0$ and $\gamma \in \Omega$ be arbitrary. Suppose $k \in \mathbb{N}$ and $j = 1, 2$ are such that $Q_k(\gamma) = P_j(\mathtt{i}, \underline{\mathtt{j}})$. Let $\mathcal{A}$ be any section of $\mathcal{I}^*$ satisfying $\eta_j^{-1}(\underline{\mathtt{j}}) \preccurlyeq \mathcal{A}$. Then

$$\sum_{\mathtt{k} \in \mathcal{A}} \beta_{\mathtt{k},j'}^{\dim_{\mathrm{A}} K_{\eta_j(\gamma),j}+\epsilon} \lesssim_{\epsilon,\gamma} 1.$$

Proof. The proof of (i) is identical to the proof given in Lemma 4.8 and similarly the proof of (iv) is identical to that of Lemma 4.10.

We now prove (ii). In order to do this, we must understand the structure of the pseudo-cylinder $P_j(\mathtt{i}, \underline{\mathtt{j}})$. Heuristically, when (for instance) $j = 1$, $P_j(\mathtt{i}, \underline{\mathtt{j}})$ is a union of cylinders which fall into one of two types: those which are tall, and those which are wide. If a cylinder is tall, we apply (i) in the opposite direction to cover it with approximate squares, and if a cylinder is wide, we group nearby cylinders together to form approximate squares. We then combine these counts using the slice dimension t_j, which is bounded above by $\dim_{\mathrm{B}} \eta_{j'}(K)$.

Write $\mathcal{A} = \eta_j^{-1}(\underline{\mathtt{j}})$ and partition $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$ where

$$\mathcal{A}_1 = \{\mathtt{k} \in \mathcal{A} : \beta_{\mathtt{ik},j'} \geq \beta_{\mathtt{i}\underline{\mathtt{j}},j}\} \qquad \text{and} \qquad \mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1.$$

First, for $\mathtt{k} \in \mathcal{A}_1$, note that $P_{j'}(\mathtt{ik}, \varnothing)$ is a wide pseudo-cylinder and we set

$$\mathcal{B}_1 = \bigcup_{\mathtt{k} \in \mathcal{A}_1} \mathcal{Q}_{j'}(\mathtt{ik}, \varnothing).$$

By applying (i), since $\beta_{\mathtt{ik},j} \approx \beta_{\mathtt{i}\underline{\mathtt{j}},j}$,

$$\#\mathcal{B}_1 = \sum_{\mathtt{k} \in \mathcal{A}_1} \#\mathcal{Q}_{j'}(\mathtt{ik}, \varnothing) \approx \sum_{\mathtt{k} \in \mathcal{A}_1} \left(\frac{\beta_{\mathtt{ik},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \tag{5.1}$$

Otherwise if $\mathtt{k} \in \mathcal{A}_2$, let $\mathtt{l}_1(\mathtt{k})$ denote the prefix of $\mathtt{k}$ of maximal length so that $\beta_{\mathtt{il}_1(\mathtt{k}),j'} \geq \beta_{\mathtt{i}\underline{\mathtt{j}},j}$. Writing $\mathtt{k} = \mathtt{l}_1(\mathtt{k})\mathtt{l}_2(\mathtt{k})$, this choice guarantees that

$$\mathcal{B}(\mathtt{k}) := P_j(\mathtt{il}_1(\mathtt{k}), \eta_j(\mathtt{l}_2(\mathtt{k})))$$

is the unique approximate square contained in $[\mathtt{i}]$ containing $[\mathtt{ik}]$. Finally, let

$$\mathcal{A}_2' = \{\mathtt{l}_1(\mathtt{k}) : \mathtt{k} \in \mathcal{A}_2\} \qquad \text{and} \qquad \mathcal{B}_2 = \{\mathcal{B}(\mathtt{k}) : \mathtt{k} \in \mathcal{A}_2\}.$$

We then note that, since $\beta_{\mathtt{il},j'} \approx \beta_{\mathtt{i}\underline{\mathtt{j}},j}$ by the choice of $\mathtt{l}_1(\mathtt{k})$,

$$\#\mathcal{B}_2 \approx \sum_{\mathtt{l} \in \mathcal{A}_2'} \left(\frac{\beta_{\mathtt{il},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \tag{5.2}$$

To conclude, observe that $\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) = \mathcal{B}_1 \cup \mathcal{B}_2$ and applying (5.1) and (5.2),

$$\begin{aligned}
\#\mathcal{Q}_j(\mathtt{i}, \underline{\mathtt{j}}) &= \#\mathcal{B}_1 + \#\mathcal{B}_2 \\
&\lesssim \sum_{\mathtt{k} \in \mathcal{A}_1} \left(\frac{\beta_{\mathtt{ik},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} + \sum_{\mathtt{l} \in \mathcal{A}_2'} \left(\frac{\beta_{\mathtt{il},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \\
&= \left(\frac{\beta_{\mathtt{i},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)} \sum_{\mathtt{k} \in \mathcal{A}_1 \cup \mathcal{A}_2'} \beta_{\mathtt{k},j'}^{\dim_{\mathrm{B}} \eta_{j'}(K)} \\
&\leq \left(\frac{\beta_{\mathtt{i},j'}}{\beta_{\mathtt{i}\underline{\mathtt{j}},j}}\right)^{\dim_{\mathrm{B}} \eta_{j'}(K)}
\end{aligned}$$

where the last line follows since $\mathcal{A}_1 \cup \mathcal{A}_2' \succcurlyeq \eta_j^{-1}(\underline{\mathtt{j}})$ is a section and $\dim_{\mathrm{B}} \eta_{j'}(K) \geq t_j(\underline{\mathtt{j}})$ where

$$\sum_{\mathtt{k} \in \mathcal{A}_1 \cup \mathcal{A}_2'} \beta_{\mathtt{k},j'}^{t_j(\underline{\mathtt{j}})} = 1.$$

Finally, we combine the bounds given in (i) and (ii) with a similar argument to the proof of Lemma 4.9 to obtain (iii). Let $\epsilon > 0$ be arbitrary and fix $\mathtt{i} \in \mathcal{I}^*$ and $j = 0, 1$ so that $0 < r \leq \beta_{\mathtt{i},j'} \leq \beta_{\mathtt{i},j}$. Write $\delta = r/\beta_{\mathtt{i},j'}$ so, recalling the proof of [Bar07, Theorem B],

$$\#\mathcal{S}(\delta) \lesssim_\epsilon (1/\delta)^{\dim_{\mathrm{B}} K+\epsilon}.$$

Now enumerate

$$\mathcal{S}(\delta) = \{Q_{1,j}, \ldots, Q_{m_j,j}\} \cup \{Q_{1,j'}, \ldots, Q_{m_{j'},j'}\}$$

where for each $z = j, j'$ and $1 \leq i \leq m_z$,

$$Q_{i,z} = P_z(\mathtt{j}_{i,z}, \underline{\mathtt{k}}_{i,z})$$

for some $\mathtt{j}_{i,z} \in \mathcal{I}^*$ and $\underline{\mathtt{k}}_{i,z} \in \eta_z(\mathcal{I}^*)$. Observe that each $P_z(\mathtt{i}\mathtt{j}_{i,z}, \underline{\mathtt{k}}_{i,z})$ is a wide pseudo-cylinder if $z = j$ and a tall pseudo-cylinder if $z = j'$. Thus we may complete the proof in the same way as Lemma 4.9, by applying (i) to the wide pseudo-cylinders and (ii) to the tall pseudo-cylinders. □

We can now prove the following formulas for the pointwise Assouad dimension.

Proposition 5.3. *Let K be a Barański carpet. Then for each $j = 1, 2$, if $\eta_j(K)$ satisfies the SSC, for all $\gamma \in \Omega_j$ and $x = \pi(\gamma)$,*

$$\dim_{\mathrm{A}}(K, x) = \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}\}$$

and

$$\max\{\dim_{\mathrm{H}} F : F \in \operatorname{Tan}(K, x)\} = \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}.$$

Furthermore,

$$\dim_{\mathrm{A}} K_{\eta_j(\gamma),j} = \lim_{m\to\infty} \sup_{n\in\mathbb{N}} \theta_{\eta_j(\gamma),j}(n, m) \leq \max_{\underline{\ell}\in\eta_j(\mathcal{I})} t_j(\underline{\ell}).$$

Proof. If $\gamma \in \Omega_j$, by definition there is a constant $\kappa \in (0, 1)$ so that

$$\frac{\beta_{\gamma|_k,j'}}{\beta_{\gamma|_k,j}} \lesssim \kappa^n.$$

In particular, there is a constant $\kappa' \in (0, 1)$ so that each maximal cylinder $[\mathtt{i}]$ contained in $Q_k(\gamma)$ has $\beta_{\mathtt{i},j'}/\beta_{\mathtt{i},j} \lesssim (\kappa')^k$, which converges to zero. Thus the same proof as given in Proposition 4.11 but instead applying Lemma 5.2 in place of the analogous bounds for Gatzouras–Lalley carpets gives that

$$\dim_{\mathrm{A}}(K, x) \leq \max\{\dim_{\mathrm{B}} K, \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}\}.$$

Similarly, the same proof as Proposition 4.7 shows that

$$\max\{\dim_{\mathrm{H}} F : F \in \operatorname{Tan}(K, x)\} = \dim_{\mathrm{B}} \eta_j(K) + \dim_{\mathrm{A}} K_{\eta_j(\gamma),j}.$$

Finally, using the same subadditivity properties of $\theta_{\eta(\gamma),j}(n,m)$ established at the beginning of the proof of Theorem 3.7,

$$\lim_{m\to\infty}\sup_{n\in\mathbb{N}}\theta_{\eta_j(\gamma),j}(n,m)\leq\max_{\underline{\ell}\in\eta_j(\mathcal{I})}t_j(\underline{\ell}).$$

as required. □

5.3. Barański carpets with few large tangents. In contrast to Gatzouras–Lalley carpets, the analogue of Theorem 4.14 need not hold for Barański carpets. We first give a precise characterization of when a Barański carpet has few large tangents. Fix the definitions of t_j and d_j from Proposition 5.1.

Theorem 5.4. *Let K be a Barański carpet such that $\eta_j(K)$ satisfies the SSC and $\Omega_j\neq\emptyset$ for $j=1,2$. Suppose for one of $j=1,2$, $d_j<d_{j'}$ and $\dim_{\mathrm{B}}\eta_j(K)+t_j>\dim_{\mathrm{B}}\eta_{j'}(K)+t_{j'}$. Then*

$$\dim_{\mathrm{H}}\{x\in K:\dim_{\mathrm{A}}(K,x)=\dim_{\mathrm{A}}K\}<\dim_{\mathrm{H}}K.$$

Proof. Suppose $d_1<d_2$ and $\dim_{\mathrm{B}}\eta_1(K)+t_1>\dim_{\mathrm{B}}\eta_2(K)+t_2$ (the opposite case follows analogously). By Proposition 5.1, $\dim_{\mathrm{H}}K=d_2$ and $\dim_{\mathrm{A}}K=\dim_{\mathrm{B}}\eta_1(K)+t_1$. In particular, by Proposition 5.3, if $\dim_{\mathrm{A}}(K,x)=\dim_{\mathrm{A}}K=\dim_{\mathrm{B}}\eta_1(K)+t_1$, then necessarily $x=\pi(\gamma)$ where $\gamma\in\Omega_0\cup\Omega_1$. But $\dim_{\mathrm{H}}\pi(\Omega_0\cup\Omega_1)=d_1<d_2=\dim_{\mathrm{H}}K$, as required. □

Remark 5.5. In the context of Theorem 5.4, one can in fact prove that the following are equivalent:
(i) $\dim_{\mathrm{H}}\{x\in K:\dim_{\mathrm{A}}(K,x)=\dim_{\mathrm{A}}K\}<\dim_{\mathrm{H}}K$.
(ii) $\dim_{\mathrm{H}}\{x\in K:\exists F\in\mathrm{Tan}(K,x)$ such that $\dim_{\mathrm{H}}F=\dim_{\mathrm{A}}K\}<\dim_{\mathrm{H}}K$.
(iii) For one of $j=1,2$, $d_j<d_{j'}$ and $\dim_{\mathrm{B}}\eta_j(K)+t_j>\dim_{\mathrm{B}}\eta_{j'}(K)+t_{j'}$.
Such a proof follows similarly to the Gatzouras–Lalley case with appropriate modifications to restrict attention only to the family Ω_1 or Ω_2. The only additional observation required is that [FJS10, Lemma 4.3] also holds in the Barański case and the uniform subsystem can be chosen so the maps are contracting strictly in direction j and the dimension of the corresponding attractor is arbitrarily close to d_j.

In particular, if one of the above equivalent conditions hold and without loss of generality $d_1>d_2$ and $\dim_{\mathrm{B}}\eta_1(K)+t_1<\dim_{\mathrm{B}}\eta_2(K)+t_2$, then the Hausdorff dimension of the level set $\varphi(\alpha)=\dim_{\mathrm{H}}\{x\in K:\dim_{\mathrm{A}}(K,x)=\alpha\}$ is given by the piecewise formula

$$\varphi(\alpha)=\begin{cases}\dim_{\mathrm{H}}K & :\dim_{\mathrm{B}}K\leq\alpha\leq\dim_{\mathrm{B}}\eta_1(K)+t_1\\ d_2 & :\dim_{\mathrm{B}}\eta_1(K)+t_1<\alpha\leq\dim_{\mathrm{A}}K.\end{cases}$$

We leave the remaining details to the curious reader.

With Theorem 5.4 in hand, we can now give an explicit example of a Barański carpet which has few large tangents.

Corollary 5.6. *There is a Barański carpet K such that*

$$\dim_{\mathrm{H}}\{x \in K : \dim_{\mathrm{A}}(K,x) = \dim_{\mathrm{A}} K\} < \dim_{\mathrm{H}} K.$$

Proof. Fix some $\delta \in [0,1/6)$ and define parameters $\beta = 1/4-\delta$, $\alpha_1 = 1/3-\delta$, and $\alpha_2 = 1/6-\delta$. Now define the families of maps

$$\begin{aligned}\Phi_1 &= \{(x,y)\mapsto(\alpha_1 x, \beta y + i\beta) : i = 0,\ldots,3\}\\ \Phi_{2,a} &= \{(x,y)\mapsto(\alpha_2 x + \alpha_1 + j\alpha_2, \beta y + i\beta) : j = 0,1 \text{ and } i = 0,1\}\\ \Phi_{2,b} &= \{(x,y)\mapsto(\alpha_2 x + \alpha_1 + j\alpha_2, \beta y + i\beta) : j = 3,4 \text{ and } i = 2,3\}\end{aligned}$$

and then set

$$\Phi_2 = \Phi_{2,a}\cup\Phi_{2,b} \qquad \text{and} \qquad \Phi = \Phi_1\cup\Phi_{2,a}\cup\Phi_{2,b}.$$

We abuse notation and use functions and indices interchangeably. Now Φ is a Barański IFS with three columns corresponding to Φ_1, $\Phi_{2,a}$, and $\Phi_{2,b}$. This carpet is conjugate to the carpet generated by the maps depicted in Figure 2b. Note that if $\delta > 0$, both projected IFSs satisfy the SSC.

We now simplify the dimensional expression in Proposition 5.1 (ii) for our specific system. First, for $\boldsymbol{w}\in\mathcal{P}$, set $p = \sum_{i\in\Phi_2}\boldsymbol{w}_i$. Note that $\chi_1(\boldsymbol{w}) = -p\log\alpha_2 - (1-p)\log\alpha_1$ and $\chi_2(\boldsymbol{w}) = -\log\beta$ depend only on p. But since entropy and projected entropy are maximized uniquely by uniform vectors, defining the vector $\boldsymbol{z}(p)\in\mathcal{P}$ by

$$\boldsymbol{z}(p)_i = \begin{cases}\frac{1-p}{4} : i \in \Phi_1\\ \frac{p}{8} : i\in\Phi_2\end{cases}$$

we necessarily have

$$\begin{aligned}\frac{H(\eta_1(\boldsymbol{w}))}{\chi_1(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_1(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} &\leq \frac{H(\eta_1(\boldsymbol{z}(p)))}{\chi_1(\boldsymbol{z}(p))} + \frac{H(\boldsymbol{z}(p)) - H(\eta_1(\boldsymbol{z}(p)))}{\chi_2(\boldsymbol{z}(p))}\\ &= \frac{-p\log p - (1-p)\log(1-p)}{-p\log\alpha_2 - (1-p)\log\alpha_1}\\ &=: D_1(p)\end{aligned}$$

and

$$\begin{aligned}\frac{H(\eta_2(\boldsymbol{w}))}{\chi_2(\boldsymbol{w})} + \frac{H(\boldsymbol{w}) - H(\eta_2(\boldsymbol{w}))}{\chi_1(\boldsymbol{w})} &\leq \frac{H(\eta_2(\boldsymbol{z}(p)))}{\chi_2(\boldsymbol{z}(p))} + \frac{H(\boldsymbol{z}(p)) - H(\eta_2(\boldsymbol{z}(p)))}{\chi_1(\boldsymbol{z}(p))}\\ &= \frac{\log 4}{-\log\beta} + \frac{-p\log p - (1-p)\log(1-p) - \log 4}{-p\log\alpha_2 - (1-p)\log\alpha_1}\\ &=: D_2(p).\end{aligned}$$

Moreover, writing $p_0 = \frac{\log\alpha_1 - \log\beta}{\log\alpha_1 - \log\alpha_2}$, $\boldsymbol{z}(p)\in\mathcal{P}_1$ if and only if $p\in[0,p_0]$ and $\boldsymbol{z}(p)\in\mathcal{P}_2$ if and only if $p\in[p_0,1]$. We thus observe that

$$\dim_{\mathrm{H}} K = \sup_{p\in[0,1]} D(p) \qquad \text{where} \qquad D(p) = \begin{cases} D_1(p) & : 0\leq p\leq p_0\\ D_2(p) & : p_0\leq p\leq 1\end{cases}.$$

Now, a manual computation directly shows that, substituting $\delta = 0$,

$$\sup_{p\in[0,1]} D_1(p) \approx 0.489536 \qquad\qquad \sup_{p\in[0,1]} D_2(p) \approx 0.529533$$

and moreover the maximum of $D_2(p)$ is attained at a value $p_2 \in (p_0, 1)$. Thus for all δ sufficiently close to 0, since all the respective quantities are continuous functions of δ, there is a value $p_2 \in (p_0, 1)$ so that

$$d_1 \leq \sup_{p\in[0,1]} D_1(p) < \sup_{p\in[0,1]} D(p) = D_2(p_2) = d_2.$$

(In fact, one can show that this is the case for all $\delta \in (0, 1/6)$, but this is not required for the proof.)

On the other hand, when $\delta = 0$, $t_1 = 2$ whereas $t_2 = 1+s < 2$ where $s \approx 0.72263$ is the unique solution to

$$\left(\frac{1}{3}\right)^s + 2 \cdot \left(\frac{1}{6}\right)^s = 1.$$

Thus for all δ sufficiently close to 0, the conditions for Theorem 5.4 are satisfied, as required. □

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