Visible parts and lower bounds on point-ray incidences

TUOMAS ORPONEN & ALEX RUTAR

ABSTRACT. Let $K \subset \mathbb{R}^2$ be a compact set. For $\theta \in S^1$, let $\mathrm{Vis}_\theta(K) \subset K$ be the visible part of K in direction θ. We prove that $\dim_{\mathrm{H}} \mathrm{Vis}_\theta(K) \leq \frac{3}{2}$ for $\mathcal{H}^1$ almost every $\theta \in S^1$. The previous record was $\dim_{\mathrm{H}} \mathrm{Vis}_\theta(K) \leq 11/6 \approx 1.833$, due to D. Dąbrowski.

Our main tool is a variant of a recent incidence lower bound theorem due to Cohen, Pohoata, and Zakharov where, roughly speaking, lines have been replaced by rays, and δ^ε-separated incidences are replaced by 1-separated incidences.

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1. INTRODUCTION

1.1. Visible parts and their dimensions. We start by introducing basic notation.

Notation 1.1. For $z \in \mathbb{R}^2$ and $\sigma \in [-1,1]$, let $\ell^+_{z,\sigma} := \{z + (r, \sigma r) : r \geq 0\}$ be the *ray with slope* σ *starting from* z. Similarly, let $\ell_{z,\sigma} := \{z + (r, \sigma r) : r \in \mathbb{R}\}$ be the *line with slope* σ *passing through* z. For $\omega = (z, \sigma) \in \mathbb{R}^2 \times [-1,1]$, we also write $\ell^+_\omega := \ell^+_{z,\sigma}$ and $\ell_\omega = \ell_{z,\sigma}$.

Definition 1.2 ($\mathrm{Vis}_\sigma(K)$). Let $K \subset \mathbb{R}^2$ be a compact set, and let $\sigma \in [-1,1]$. The *visible part of* K *in direction* σ is the set

$$\mathrm{Vis}_\sigma(K) := \{z \in K : K \cap \ell^+_{z,\sigma} = \{z\}\}.$$

We are interested in the following conjecture.

Conjecture 1.3 (Visibility conjecture). *Let* $K \subset \mathbb{R}^2$ *be compact. Then* $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) \leq 1$ *for* $\mathcal{H}^1$*-a.e.* $\sigma \in [-1,1]$.

More generally, the following estimate on the set of directions with exceptionally large visible part seems plausible.

Conjecture 1.4 (Exceptional set estimate for visible parts). *Let* $K \subset \mathbb{R}^2$ *be compact and* $\alpha \in [0,1]$. *Then* $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) \leq 2 - \alpha$ *for* $\mathcal{H}^\alpha$*-a.e.* $\sigma \in [-1,1]$.

Remark 1.5. Typically the visibility conjecture is stated in a form where directions are parametrised by S^1 in place of $[-1,1]$; we also used this convention in the abstract. It is easy to check that every upper bound for the variants in Conjecture 1.3 and Conjecture 1.4 translates immediately to a corresponding upper bound for the S^1-variant.

Remark 1.6. Máthé has constructed[1] for each $\alpha \in [0,1]$ a compact set $N \subset \mathbb{R}^2$ such that $\dim_{\mathrm{H}} N = 2 - \alpha$, and $\mathrm{Vis}_\theta N = N$ in an α-dimensional set of directions $\theta \in S^1$. This shows that Conjecture 1.4 would be sharp if true. Mathé initially constructed his example to show the sharpness of the main result in [DGO24+] concerning α*-Nikodým sets*.

The visibility conjecture is a well-known open problem in geometric measure theory; see, for instance, [JJMO03] [Mat04, Problem 11], [AJJ+12, (1.3)], [FF13, Conjecture 1.3] or [Fal26+, Section 7]. The conjecture in $\mathbb{R}^2$ has been resolved for some special families of sets such as quasi-circles and some classes of self-similar and self-affine sets (see [AJJ+12; FF13; JJMO03; JJSW22; Ros21]). There is also a strong partial result for planar continua due to O'Neil [ONe07]. In [JJN05], it is shown that if $s \in (1,2]$, and $K \subset \mathbb{R}^2$ is a compact set with $\mathcal{H}^s(K) < \infty$, then $\mathcal{H}^s(\mathrm{Vis}_\sigma(K)) = 0$ for a.e. $\sigma \in [-1,1]$. It however remains open (under the same assumptions) whether $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) < s$ for almost all $\sigma \in [-1,1]$.

The best previous result for general compact sets is due to Dąbrowski [Dąb24], who showed that $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) \leq 11/6 \approx 1.833$ for a.e. $\sigma \in [-1,1]$. Weaker bounds in this generality were earlier obtained by Matheus and the first author; see [Mat21; Orp23].

[1] The details of Máthé's construction are forthcoming, but a sketch can be found at https://www.tu-chemnitz.de/mathematik/stochastik/fgs7/slides/Andras-Mathe.pdf.

Dąbrowski also proved that if $K \subset \mathbb{R}^2$ is compact and Ahlfors s-regular with $s \in (1,2]$, then $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) \le s - a(s-1)$ for a.e. $\sigma \in [-1,1]$, where $a > 0.183$ is absolute. Finally, we mention that many of the previous partial results also extend to higher dimensions.

Our main result is the following improvement for general compact sets in $\mathbb{R}^2$.

Theorem A. *Let $K \subset \mathbb{R}^2$ be compact and $\alpha \in [0,1]$. Then $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) \le 2 - \frac{\alpha}{2}$ for $\mathcal{H}^\alpha$-a.e. $\sigma \in [-1,1]$. In particular, $\dim_{\mathrm{H}} \mathrm{Vis}_\sigma(K) \le \frac{3}{2}$ for $\mathcal{H}^1$-a.e. $\sigma \in [-1,1]$.*

We will prove Theorem A by reducing it to a δ-discretised problem concerning lower bounds on "ray incidences", discussed in the next section.

1.2. Lower bounds on ray incidences. In [CPZ25], Cohen, Pohoata and Zakharov proved a beautiful result guaranteeing many δ-discretised point-line incidences. The δ-discretised result behind Theorem A is a variant of Cohen, Pohoata and Zakharov's theorem. We now proceed to state both the original result, and the variant we will prove in this paper.

Definition 1.7 (Phase space and Frostman $(\delta, \alpha, \beta, C)$-sets). Let $\Omega := [-1,1]^3$ be the *phase space*. For $u, v, w \in (0,1]$ and $(a, b, \sigma) \in \Omega$ we define the *phase space rectangle*

$$\mathbf{R}_{u \times v \times w}(a, b, \sigma) := \{(a + r_1, b + \sigma r_1 + r_2, \sigma + r_3) : (r_1, r_2, r_3) \in [-u, u] \times [-v, v] \times [-w, w]\}.$$

Let $\alpha, \beta \ge 0$, $\delta \in (0,1]$, and $C > 0$. A finite set $\mathbf{X} \subset \Omega$ is called a *Frostman $(\delta, \alpha, \beta, C)$-set* if $\mathbf{X}$ is non-empty, and

$$|\mathbf{X} \cap \mathbf{R}_{u \times uw \times w}(a, b, \sigma)| \le C u^\alpha w^\beta |\mathbf{X}|, \qquad (a, b, \sigma) \in \mathbb{R}^3,\ u, w \in (0,1],\ uw \ge \delta. \tag{1.1}$$

Notation 1.8. For $\omega = (a, b, c) \in \Omega$, we write $z_\omega := (a, b)$. We also remind the reader of the notation ℓ_ω and ℓ_ω^+ defined already in Notation 1.1. We will also write $P[\mathbf{X}] := \{z_\omega : \omega \in \mathbf{X}\}$ for the "base point projection" of $\mathbf{X}$.

Definition 1.9 (Line and ray incidences). Let $\mathbf{X} \subset \Omega$ and $\delta \in (0,1]$. For $\mathbf{A}, \mathbf{B} \subset \mathbf{X}$, we define the *line incidences*

$$\mathcal{I}_\delta(\mathbf{A}, \mathbf{B}) := |\{(\omega_1, \omega_2) \in \mathbf{A} \times \mathbf{B} : \mathrm{dist}(z_{\omega_2}, \ell_{\omega_1}) \le \delta\}|.$$

We also define the *ray incidences* by replacing ℓ_{ω_1} by $\ell_{\omega_1}^+$:

$$\mathcal{I}_\delta^+(\mathbf{A}, \mathbf{B}) := |\{(\omega_1, \omega_2) \in \mathbf{A} \times \mathbf{B} : \mathrm{dist}(z_{\omega_2}, \ell_{\omega_1}^+) \le \delta\}|.$$

We abbreviate $\mathcal{I}_\delta(\mathbf{A}, \mathbf{A}) =: \mathcal{I}_\delta(\mathbf{A})$ and $\mathcal{I}_\delta^+(\mathbf{A}, \mathbf{A}) =: \mathcal{I}_\delta^+(\mathbf{A})$.

Here is Cohen, Pohoata and Zakharov's theorem [CPZ25, Theorem 1.9] in its original form:

Theorem 1.10 ([CPZ25]). *Let $\alpha, \beta \in (1,2]$ with $\alpha + \beta > 3$, and let $\varepsilon > 0$. Then, there exist $\eta = \eta(\alpha, \beta, \varepsilon) > 0$ and $\delta_0 = \delta_0(\alpha, \beta, \varepsilon) > 0$ such that the following holds for all $\delta \in 2^{-\mathbb{N}} \cap (0, \delta_0]$:*

Let $\mathbf{X} \subset \Omega$ be a Frostman $(\delta, \alpha, \beta, \delta^{-\eta})$-set. Then,

$$\mathcal{I}_\delta(\mathbf{X}) \ge \delta^{1+\varepsilon} |\mathbf{X}|^2. \tag{1.2}$$

Roughly speaking, Theorem 1.10 says that a (δ, α, β)-Frostman set $\mathbf{X} \subset \Omega$ spans as many δ-discretised point-line incidences as it would if the points and lines were placed "at random". Starting from a counter assumption to our main result, Theorem A, it is relatively straightforward to construct an appropriate Frostman $(\delta, s, s+\alpha-1, C)$-set $\mathbf{X} \subset \Omega$ to which one is tempted to apply Theorem 1.10. The construction of $\mathbf{X}$ is accomplished in §2.2. However, a direct application of Theorem 1.10 fails to prove Theorem A for two distinct reasons:

1. We need a lower bound for $\mathcal{I}_\delta^+(\mathbf{X})$ in place of $\mathcal{I}_\delta(\mathbf{X})$.
2. The δ^ε-factor in (1.2) is unacceptable for us. Roughly speaking, this lower bound could be used to prove the existence of two δ^ε-separated incidences. For Theorem A, we need to upgrade δ^ε to a constant independent of δ. However, this conclusion may fail for Frostman $(\delta, \alpha, \beta, \delta^{-\eta})$-sets, as in Theorem 1.10, where the Frostman constant depends on δ.

In the application arising from Theorem A, fortunately, the Frostman constant of $\mathbf{X}$ is independent of δ. So, we need to establish a variant of Theorem 1.10 where both the hypothesis and conclusion are stronger. This is Theorem B below.

Theorem B. *Let $\alpha, \beta \in (1, 2]$ with $\alpha + \beta > 3$. Then, there exists $\kappa = \kappa(\alpha, \beta) \geq 1$ such that the following holds for all $C \geq 1$ and all $\delta \in 2^{-\mathbb{N}} \cap (0, \kappa^{-1}C^{-\kappa}]$:*

Let $\mathbf{X} \subset \Omega$ be a Frostman $(\delta, \alpha, \beta, C)$-set. Then there exist $\mathbf{A}, \mathbf{B} \subset \mathbf{X}$ such that

$$\operatorname{dist}(P[\mathbf{A}], P[\mathbf{B}]) \gtrsim_{\alpha,\beta} C^{-\kappa} \quad \text{and} \quad \mathcal{I}_\delta^+(\mathbf{A}, \mathbf{B}) \gtrsim_{\alpha,\beta} C^{-\kappa} \cdot \delta |\mathbf{X}|^2.$$

In particular, there exists a pair $(\omega_1, \omega_2) \in \mathbf{A} \times \mathbf{B}$ such that

$$|z_{\omega_1} - z_{\omega_2}| \gtrsim_{\alpha,\beta} C^{-\kappa} \quad \text{and} \quad \operatorname{dist}(z_{\omega_2}, \ell_{\omega_1}^+) \leq \delta. \tag{1.3}$$

Remark 1.11. In particular, Theorem B implies that $\mathcal{I}_\delta^+(\mathbf{X}) \gtrsim_{\alpha,\beta,C} \delta |\mathbf{X}|^2$. It is conceivable (but non-trivial) that this alone guarantees c-separated incidences (with $c \sim_{\alpha,\beta,C} 1$). In any case, proving the stated "bi-partite" version of Theorem B poses no additional difficulties, and gives the ρ-separation of the incidences for free.

Perhaps surprisingly, we are able to formally deduce Theorem B from the following incidence lower bound for rays which still has the δ^ε-factor in the conclusion. The statement is the same as [CPZ25, Theorem 1.9], except with rays in place of lines.

Theorem C. *Let $\alpha, \beta \in (1, 2]$ with $\alpha + \beta > 3$, and let $\varepsilon > 0$. Then, there exist $\eta = \eta(\alpha, \beta, \varepsilon) > 0$ and $\delta_0 = \delta_0(\alpha, \beta, \varepsilon) > 0$ such that the following holds for all $\delta \in 2^{-\mathbb{N}} \cap (0, \delta_0]$:*

Let $\mathbf{X} \subset \Omega$ be a Frostman $(\delta, \alpha, \beta, \delta^{-\eta})$-set. Then,

$$\mathcal{I}_\delta^+(\mathbf{X}) \geq \delta^{1+\varepsilon} |\mathbf{X}|^2.$$

1.3. Outline of the paper. As mentioned earlier, the main result on visible parts (Theorem A) is deduced from Theorem B. The first step is to find an appropriate δ-discretised version of Theorem A. This is accomplished in §2. The proof of the δ-discretised counterpart is then completed in §2.2 by identifying an appropriate Frostman set $\mathbf{X} \subset \Omega$, and applying Theorem B.

Most of the paper is occupied by the proof of Theorem B. There are two distinct steps. The first one is to obtain the (strictly stronger) version of Theorem 1.10 where $\mathcal{I}_\delta(\mathbf{X})$

is simply replaced by $\mathcal{I}_\delta^+(\mathbf{X})$, as stated in Theorem C. For those familiar with Cohen, Pohoata, and Zakharov's proof of Theorem 1.10, we mention that the least trivial step in the proof of our extension is to prove a version of the "high-low lemma" for rays in place of lines; see Theorem 3.5. In the corresponding proof of the high-low lemma for lines ([CPZ25, Theorem 1.7]), a certain orthogonality property for pairs of lines is used, which is no longer true for (all) pairs of rays. The solution is to examine the pairs of rays where the orthogonality relation may fail, and prove that there are not too many of them.

Perhaps a little surprisingly, Theorem B can be formally deduced from Theorem C. The idea is to first apply Theorem C at a suitable intermediate scale $\Delta = \Delta(\alpha, \beta, C) > 0$. Then, one applies the high-low inequality (for rays) to show that many Δ-separated Δ-discretised point-ray incidences also guarantee many Δ-separated δ-discretised point-ray incidences with $\delta \ll \Delta$.

To summarise, the logical structure of the paper is as follows:

$$\text{Theorem C} \implies \text{Theorem B} \implies \text{Proposition 2.1} \implies \text{Theorem A}.$$

The sections follow this implication chain in reverse order. We prove Theorem C in §4; the first implication is established in §3; and the final two implications are established in §2.

Basic notation. For $r \in 2^{-\mathbb{N}}$, and $K \subset \mathbb{R}^d$ (including $K \subset \Omega = [-1,1]^3$), the notation $\mathcal{D}_r(K)$ stands for the dyadic cubes of side-length r which intersect K. For $K \subset \mathbb{R}^d$, and $r > 0$, the notation $[K]_r$ stands for the open r-neighbourhood of K. For finite sets $P \subset \mathbb{R}^d$ (or commonly also $\mathbf{X} \subset \Omega$), the notation $|P|$ stands for the cardinality of P. For $z_1, z_2 \in \mathbb{R}^d$, the notation $|z_1 - z_2|$ means the Euclidean distance of z_1, z_2.

For $A, B \geq 0$, the notation $A \lesssim_p B$ means that $A \leq CB$, where $C > 0$ is a constant depending only on p. If there is no "p" visible, then C is absolute. For $A, B, \delta > 0$, the notation $A \lessapprox_\delta B$ means the same as $A \leq C(\log \frac{1}{\delta})^C B$, where $C > 0$ is absolute.

The notation $\mathcal{H}^s_\infty$ stands for s-dimensional Hausdorff content.

2. REDUCTION OF THE VISIBILITY CONJECTURE TO AN INCIDENCE LOWER BOUND

We explain how to reduce Theorem A to the incidence lower bound stated in Theorem B.

2.1. Discretisation of the visibility conjecture. We first reduce Theorem A to the following δ-discretised counterpart:

Proposition 2.1. *For every $\alpha \in (0,1]$ and $s > 2 - \frac{\alpha}{2}$ and $C > 0$ there exists $\eta = \eta(C, s, \alpha) > 0$ and $\delta_0 = \delta_0(C, s, \alpha) > 0$ such that the following holds for all $\delta \in 2^{-\mathbb{N}} \cap (0, \delta_0]$:*

Let $\Sigma \subset \delta \cdot \mathbb{Z} \cap [-1,1]$ be a non-empty Frostman (δ, α, C)-set. For each $\sigma \in \Sigma$, let $P_\sigma \subset (\delta \cdot \mathbb{Z})^2 \cap [-1,1]^2$ be a non-empty Frostman (δ, s, C)-set. Write $P := \bigcup_{\sigma \in \Sigma} P_\sigma$. Then, there exist $\sigma \in \Sigma$ and $z \in P_\sigma$ such that

$$\operatorname{diam}(P \cap [\ell^+_{z,\sigma}]_\delta) \geq \eta. \tag{2.1}$$

We now deduce Theorem A from Proposition 2.1. The argument is nearly the same as the proof of the implication between [DGO24+, Theorem 3.2] and [DGO24+, Theorem 3.1], but we give all the details for the reader's convenience.

Proof (of Theorem A assuming Proposition 2.1). The case $\alpha = 0$ of Theorem A is trivial, so we may assume that $\alpha \in (0,1]$. We make a counter assumption: there exists a compact set $K \subset [0,1]^2$, $s \in (2 - \frac{\alpha}{2}, 2]$, and $\kappa > 0$, such that

$$\mathcal{H}^{\alpha}_{\infty}(\{\sigma \in [-1,1] : \mathcal{H}^{s}_{\infty}(\mathrm{Vis}_{\sigma}(K)) > \kappa\}) > \kappa. \tag{2.2}$$

Let $C \geq 1$ be an absolute constant, and let $\eta := \eta(C\kappa^{-1}, s, \alpha) > 0$ be the constant provided by Proposition 2.1. In fact, C is the constant from the following version of Frostman's lemma: if $\delta \in 2^{-\mathbb{N}}$, $s \in (1,2]$, and $H \subset [0,1]^2$, then $\mathcal{D}_{\delta}(H)$ contains a non-empty Frostman $(\delta, s, C/\mathcal{H}^{s}_{\infty}(H))$-set. For a proof of this statement, see [Orp25, Proposition 3.9].

For $\sigma \in [-1,1]$, let

$$\mathrm{Vis}^{\eta}_{\sigma}(K) := \{x \in K : K \cap \ell^{+}_{x,\sigma} \subset B(x, \eta/8)\}.$$

Recall that $B(x,\eta/8)$ is an open disc, so if $x \in \mathrm{Vis}^{\eta}_{\sigma}(K)$ and $y \in K$ with $y \in \ell^{+}_{x,\sigma}$, then $|x - y| < \eta/8$. Clearly $\mathrm{Vis}_{\sigma}(K) \subset \mathrm{Vis}^{\eta}_{\sigma}(K)$, so (2.2) implies

$$\mathcal{H}^{\alpha}_{\infty}(\{\sigma \in [-1,1] : \mathcal{H}^{s}_{\infty}(\mathrm{Vis}^{\eta}_{\sigma}(K)) > \kappa\}) > \kappa. \tag{2.3}$$

For every $\varepsilon > 0$, we further define

$$\mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K) := \{x \in K : K \cap [\ell^{+}_{x,\sigma}]_{\varepsilon} \subset B(x,\eta/8)\}, \tag{2.4}$$

where $[\ell^{+}_{x,\sigma}]_{\varepsilon}$ stands for the open ε-neighbourhood of the ray $\ell^{+}_{x,\sigma}$. We claim that

$$\mathrm{Vis}^{\eta}_{\sigma}(K) = \bigcup_{\varepsilon > 0} \mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K), \qquad \sigma \in [-1,1]. \tag{2.5}$$

The inclusion $\mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K) \subset \mathrm{Vis}^{\eta}_{\sigma}(K)$ clearly holds for all $\varepsilon > 0$, so it suffices to prove the other inclusion. Fix $x \in \mathrm{Vis}^{\eta}_{\sigma}(K)$, so $K \cap \ell^{+}_{x,\sigma} \subset B(x,\eta/8)$. If $x \notin \bigcup_{\varepsilon > 0} \mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K)$, then for every $\varepsilon > 0$ there exists $y_{\varepsilon} \in K \cap [\ell^{+}_{x,\sigma}]_{\varepsilon} \setminus B(x,\eta/8)$. Since K is compact, we may find a sequence $\{\varepsilon_n\}$ such that $y_{\varepsilon_n} \to y \in K \cap \ell^{+}_{x,\sigma}$. Since $|x - y_{\varepsilon_n}| \geq \eta/8$, we also have $|x - y| \geq \eta/8$. This contradicts the definition of $x \in \mathrm{Vis}^{\eta}_{\sigma}(K)$ and proves (2.5).

As $\varepsilon \searrow 0$, the sets $\mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K)$ increase to $\mathrm{Vis}^{\eta}_{\sigma}(K)$. Therefore, if $\mathcal{H}^{s}_{\infty}(\mathrm{Vis}^{\eta}_{\sigma}(K)) > \kappa$, Davies' increasing sets lemma [Dav70, Theorem 4] implies that also $\mathcal{H}^{s}_{\infty}(\mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K)) > \kappa$ for $\varepsilon > 0$ sufficiently small. This observation implies that also the sets $\{\sigma \in [-1,1] : \mathcal{H}^{s}_{\infty}(\mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K)) > \kappa\}$ increase to $\{\sigma \in [-1,1] : \mathcal{H}^{s}_{\infty}(\mathrm{Vis}^{\eta}_{\sigma}(K)) > \kappa\}$ as $\varepsilon \searrow 0$. By a second application of [Dav70, Theorem 4], we infer from (2.3) that

$$\mathcal{H}^{\alpha}_{\infty}(\{\sigma \in [-1,1] : \mathcal{H}^{s}_{\infty}(\mathrm{Vis}^{\eta,\varepsilon}_{\sigma}(K)) > \kappa\}) > \kappa \tag{2.6}$$

for $\varepsilon > 0$ sufficiently small.

Let $\varepsilon > 0$ be so small that (2.6) holds, and let

$$\delta \in 2^{-\mathbb{N}} \cap (0, c\min\{\varepsilon, \eta, \delta_0\}], \tag{2.7}$$

where $c \in (0, \frac{1}{16}]$ is an absolute constant to be determined, and $\delta_0 = \delta_0(C\kappa^{-1}, s, \alpha) > 0$ is the threshold provided by Proposition 2.1. Applying Frostman's lemma, let

$$\mathcal{S} := \mathcal{S}_\delta \subset \mathcal{D}_\delta(\{\sigma \in [-1,1] : \mathcal{H}^s_\infty(\mathrm{Vis}^{\eta,\varepsilon}_\sigma(K)) > \kappa\})$$

be a non-empty $(\delta, \alpha, C\kappa^{-1})$-set. Let Σ denote the left endpoints of dyadic intervals in $\mathcal{S}$.

For each $\sigma \in \Sigma$, let $\bar{\sigma}$ denote an element in the dyadic interval containing σ with $\mathcal{H}^s_\infty(\mathrm{Vis}^{\eta,\varepsilon}_{\bar{\sigma}}(K)) > \kappa$. Thus, Frostman's lemma yields a Frostman $(\delta, s, C\kappa^{-1})$-set

$$\mathcal{P}_\sigma \subset \mathcal{D}_\delta(\mathrm{Vis}^{\eta,\varepsilon}_{\bar{\sigma}}(K)).$$

We let $P_\sigma \subset \cup\mathcal{P}_\sigma$ consist of the lower left corners of the squares in $\mathcal{P}_\sigma$. Then P_σ is also a Frostman $(\delta, s, C\kappa^{-1})$-set.

Write $P := \bigcup_{\sigma\in\Sigma} P_\sigma$, as in Proposition 2.1. We now claim a contradiction to Proposition 2.1. Namely, if $\sigma \in \Sigma$ and $z \in P_\sigma$, we claim that

$$\mathrm{diam}(P \cap [\ell^+_{z,\sigma}]_\delta) \leq \eta/2. \tag{2.8}$$

This contradiction will complete the proof of Theorem A.

To prove (2.8), fix $\sigma \in \Sigma$ and $z \in P_\sigma$. By definition, this means that there exist $\bar{\sigma} \in [-1,1]$ with $|\sigma - \bar{\sigma}| \leq \delta$, and a point $x \in \mathrm{Vis}^{\eta,\varepsilon}_{\bar{\sigma}}(K)$ which shares the dyadic δ-square with z. Unwrapping the definitions further (see (2.4)),

$$K \cap [\ell^+_{x,\bar{\sigma}}]_\varepsilon \subset B(x, \eta/8).$$

We claim that this, and $\delta \leq \frac{1}{16}\min\{\varepsilon,\eta\}$, implies $[K]_{2\delta} \cap [\ell^+_{x,\bar{\sigma}}]_{\varepsilon/2} \subset B(x,\eta/4)$. Indeed, if $y \in [K]_{2\delta} \cap [\ell^+_{x,\bar{\sigma}}]_{\varepsilon/2}$, then by the triangle inequality exists a point $\bar{y} \in K \cap [\ell^+_{x,\bar{\sigma}}]_\varepsilon \subset B(x,\eta/8)$ with $|\bar{y} - y| \leq 2\delta$, and therefore $y \in B(x,\eta/4)$.

Next, we claim that $P \cap [\ell^+_{z,\sigma}]_\delta \subset [K]_{2\delta} \cap [\ell^+_{x,\bar{\sigma}}]_{\varepsilon/2}$, which implies (2.8). Clearly $P \subset [K]_{2\delta}$. Second, the inclusion $[-2,2]^2 \cap [\ell^+_{z,\sigma}]_\delta \subset [\ell^+_{x,\bar{\sigma}}]_{\varepsilon/2}$ holds if $\delta \leq c\varepsilon$ for a sufficiently small absolute constant $c > 0$, using the facts that $|z - x| \leq 2\delta$ and $|\sigma - \bar{\sigma}| \leq \delta$.

We have now proved (2.8), and consequently Theorem A. □

2.2. From the discretised visibility conjecture to incidence lower bounds. In this section we deduce Proposition 2.1 from Theorem B. Since in §2 we saw that Proposition 2.1 implies Theorem A, this completes the proof of Theorem A (modulo proving Theorem B).

Proof (of Proposition 2.1 assuming Theorem B). Using the data $\{P_\sigma\}_{\sigma\in\Sigma}$ provided in the statement of Proposition 2.1, we construct a $(\delta, s, s+\alpha-1, O(C^2))$-set $\mathbf{X} \subset \Omega$, and apply Theorem B to derive (2.1). Note that $s > 2 - \frac{\alpha}{2}$ is equivalent to $s + (s + \alpha - 1) > 3$.

First of all, to prove Proposition 2.1, we may assume that Σ is a Katz–Tao $(\delta, \alpha, 1)$-set with $|\Sigma| \gtrsim \delta^{-\alpha}/C$ and that each $P_\sigma \subset (\delta\,\mathbb{Z})^2 \cap [-1,1]^2$ is a Katz–Tao $(\delta, s, 1)$-set with $|P_\sigma| \gtrsim \delta^{-s}/C$. This is because Σ and each P_σ contains such a subset, and it suffices to prove Proposition 2.1 with these subsets in place of the original sets.

Define

$$\mathbf{X} := \{(a,b,\sigma) \in (\delta\,\mathbb{Z})^3 \cap \Omega : (a,b) \in P_\sigma\}.$$

Then, $\mathbf{X}$ is δ-separated, and

$$C^{-2}\delta^{-\alpha-s} \lesssim \sum_{\sigma\in\Sigma} |P_\sigma| = |\mathbf{X}| \lesssim \delta^{-\alpha-s}. \tag{2.9}$$

We claim that $\mathbf{X}$ is a Frostman $(\delta, s, s+\alpha-1, O(C^2))$-set in the sense of Definition 1.7. To see this, fix $u, w \in (0,1]$ with $uw \geq \delta$ and $(a,b,\sigma) \in \Omega$. It suffices to show that

$$|\mathbf{X} \cap \mathbf{R}_{uw\times uw\times w}(a,b,\sigma)| \lesssim w^\alpha (uw)^s \cdot \delta^{-\alpha-s} \overset{(2.9)}{\lesssim} C^2 w^\alpha (uw)^s |\mathbf{X}|. \tag{2.10}$$

This is because the rectangle $\mathbf{R}_{u\times uw\times w}(a,b,\sigma)$ appearing in Definition 1.7 can be covered by $\lesssim w^{-1}$ rectangles of the form $\mathbf{R}_{uw\times uw\times w}(a',b,\sigma)$.

To prove (2.10), note that if $(a',b',\sigma') \in \mathbf{X}$ lies in the rectangle

$$\mathbf{R}_{uw\times uw\times w}(a,b,\sigma) = \{(a+r_1, b+\sigma r_1 + r_2, \sigma + r_3) : (r_1,r_2,r_3) \in [-uw,uw]^2 \times [-w,w]\},$$

then $(a',b') \in B((a,b),2uw) =: B_{uw} \subset \mathbb{R}^2$ and $\sigma' \in B(\sigma,w) \cap [-1,1] =: I$. Therefore, recalling the definition of $\mathbf{X}$,

$$\begin{aligned} |\mathbf{X} \cap \mathbf{R}_{uw\times uw\times w}(a,b,\sigma)| &\leq |\{(a',b',\sigma') \in B_{uw} \times I : (a',b') \in P_{\sigma'}\}| \\ &\leq \sum_{\sigma'\in\Sigma\cap I} |P_{\sigma'} \cap B_{uw}| \lesssim (w/\delta)^\alpha \cdot (uw/\delta)^s = w^\alpha (uw)^s \cdot \delta^{-\alpha-s}. \end{aligned}$$

This completes the proof of (2.10), and shows that $\mathbf{X}$ is a Frostman $(\delta, s, s+\alpha-1, C')$-set with $C' \lesssim C^2$.

To complete the proof of Proposition 2.1, apply Theorem B with parameters $\alpha, s+\alpha-1$ and C' to obtain a constant $\kappa = \kappa(\alpha, s, C') \geq 1$. Then, for $\rho \sim_{\alpha,s} (C')^{-\kappa}$, Theorem B gives a pair $(z_1,\sigma_1),(z_2,\sigma_2) \in \mathbf{X}$ such that $|z_1 - z_2| \geq \rho$ and $\mathrm{dist}(z_2, \ell^+_{z_1,\sigma_1}) \leq \delta$.

Finally, note that $\mathrm{dist}(z_2, \ell^+_{z_1,\sigma_1}) \leq \delta$ implies $\{z_1,z_2\} \subset P \cap [\ell^+_{z_1,\sigma_1}]_\delta$. Therefore $\mathrm{diam}(P \cap [\ell^+_{z_1,\sigma_1}]_\delta) \geq |z_1 - z_2| \geq \rho$. This proves (2.1) with $\eta = \rho$. □

3. INCIDENCE LOWER BOUND FOR RAYS WITH POLYNOMIAL CONSTANT DEPENDENCE

We now embark on proving Theorem B. In this section, we prove Theorem B conditional on Theorem C. We defer the proof of Theorem C until §4.

3.1. Bi-partisation at an intermediate scale. The plan is as follows: we will apply Theorem C at a large intermediate scale Δ, and then use Theorem 3.5 to guarantee many incidences at scale δ.

We start by recording a well-known lemma on finding large bi-partite subgraphs. The lemma is due to Erdős [Erd65], but the original formulation concerns undirected graphs, while we need the result for directed graphs. This variant is also well-known, but we repeat the short proof and, with no extra effort, record a weighted generalisation. The proof is from [AS16, Theorem 2.2.1], where the result is also stated for undirected and unweighted graphs, with constant "$\frac{1}{2}$" instead of "$\frac{1}{4}$".

Lemma 3.1. *Let $G = (V, E)$ be a directed graph without loops, and let $w\colon E \to (0,\infty)$ be a weight function. Then there exists a bi-partite subgraph $(V_1 \cup V_2, \bar{E})$ with $\sum_{e\in\bar{E}} w(e) \geq \frac{1}{4}\sum_{e\in E} w(e)$, where $V_1, V_2 \subset V$ are disjoint, and*

$$\bar{E} = \{(v_1, v_2) \in E : v_1 \in V_1,\ v_2 \in V_2\}. \tag{3.1}$$

Proof. Include a vertex $v \in V$ to V_1 with probability $\frac{1}{2}$, and let $V_2 := V \setminus V_1$. For $e = (v_1, v_2) \in E$, let X_e be the random variable which takes value $w(e)$ if and only if $v_1 \in V_1$ and $v_2 \in V_2$. Evidently $\mathbb{E}[X_e] = w(e)/4$, and therefore

$$\mathbb{E}\Big[\sum_{e\in E} X_e\Big] = \tfrac{1}{4}\sum_{e\in E} w(e).$$

It follows that there exists a random set V_1 such that $\sum_{e\in E} X_e \geq \frac{1}{4}\sum_{e\in E} w(e)$. Now, with $\bar{E}$ as in (3.1), it holds $\sum_{e\in\bar{E}} w(e) \geq \frac{1}{4}\sum_{e\in E} w(e)$. □

Definition 3.2. We say that sets $\mathbf{A}, \mathbf{B} \subset \Omega$ are *spatially r-separated* if $\operatorname{dist}(P[\mathbf{A}], P[\mathbf{B}]) \geq r$.

The following lemma is the key ingredient in the proof of Theorem B. Observe that the sets $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{X}$ are δ-separated so the net effect is that the incidences at scale 3Δ are counted with multiplicity.

Lemma 3.3. *For every $\alpha, \beta \in (1, 2]$ with $\alpha + \beta > 3$ and $\varepsilon > 0$, there exist $\eta = \eta(\alpha, \beta, \varepsilon) > 0$ and $\Delta_0 = \Delta_0(\alpha, \beta, \varepsilon) > 0$ such that the following holds for all $\delta, \Delta \in 2^{-\mathbb{N}} \cap (0, \Delta_0]$ with $\delta \leq \Delta$:*

Let $\mathbf{X} \subset \Omega$ be a Frostman $(\delta, \alpha, \beta, \Delta^{-\eta})$-set. Then, there exist spatially Δ-separated subsets $\mathbf{A}, \mathbf{B} \subset \mathbf{X}$ such that

$$\mathcal{I}^{+}_{\Delta}(\mathbf{A}, \mathbf{B}) \geq \Delta^{1+\varepsilon}|\mathbf{X}|^2. \tag{3.2}$$

Proof. We start by fixing parameters. We may assume that $\varepsilon \in (0, (\alpha-1)/2]$, since the statement gets stronger when ε decreases. Next, let $\eta > 0$ be so small that

$$2\eta < (\alpha - 1)/2 \leq \alpha - 1 - \varepsilon. \tag{3.3}$$

Additionally, assume that $2\eta \leq \eta_0(\alpha, \beta, \varepsilon/2)$, where $\eta_0(\alpha, \beta, \varepsilon/2) > 0$ is the constant given by Theorem C applied with parameters $\alpha, \beta, \varepsilon/2$. Let $\Delta_0 = \Delta_0(\alpha, \beta, \varepsilon/2) > 0$ be the scale threshold given by Theorem C, and assume that $\delta, \Delta \in 2^{-\mathbb{N}} \cap (0, \Delta_0]$ with $\delta \leq \Delta$. Finally, we will assume that Δ_0 is small in terms of ε, η, so that various absolute constants appearing in the proof are bounded from above by $\Delta_0^{-\eta}$ and $\Delta_0^{-\varepsilon/2}$.

We start by replacing $\mathbf{X}$ by a subset with spatial separation. Write $\mathcal{Q} := \mathcal{D}_{\Delta}([-1,1)^2)$. Split $\mathcal{Q}$ into 4 sub-families $\mathcal{Q}_1, \ldots, \mathcal{Q}_4$ such that $\operatorname{dist}(Q, Q') \geq \Delta$ for all $i = 1, \ldots, 4$ and $Q, Q' \in \mathcal{Q}_i$ with $Q \neq Q'$. Let $\mathbf{X}_j := \{\omega \in \mathbf{X} : z_\omega \in \cup\mathcal{Q}_j\}$. Then there exists $j \in \{1, \ldots, 4\}$ such that $|\mathbf{X}_j| \geq \frac{1}{4}|\mathbf{X}|$. Now $\mathbf{X}_j$ satisfies the same hypotheses as does $\mathbf{X}$ with constant "2η", and additionally: *the family $\mathcal{Q}_j := \mathcal{D}_{\Delta}(P[\mathbf{X}_j])$ is Δ-separated*. To simplify notation, we now assume that $\mathbf{X}$ originally has this property, and we write $\mathcal{Q} := \mathcal{D}_{\Delta}(P[\mathbf{X}])$.

Recall that $2\eta \leq \eta_0(\alpha, \beta, \varepsilon/2)$, $\Delta \leq \Delta_0(\alpha, \beta, \varepsilon/2)$, and note that $\mathbf{X}$ is a $(\Delta, \alpha, \beta, \Delta^{-2\eta})$-set (simply because $\delta \leq \Delta$). Theorem C applied at scale Δ now gives

$$\mathcal{I}^{+}_{\Delta}(\mathbf{X}) \geq \Delta^{1+\varepsilon/2}|\mathbf{X}|^2. \tag{3.4}$$

Now we proceed to find the sets $\mathbf{A}, \mathbf{B} \subset \mathbf{X}$ in (3.2). For each pair $(Q_1, Q_2) \in \mathcal{Q} \times \mathcal{Q}$ (including diagonal pairs (Q, Q)), define

$$m_{Q_1,Q_2} := |\{(\omega_1, \omega_2) \in \mathbf{X}^2 : z_{\omega_1} \in Q_1,\ z_{\omega_2} \in Q_2 \text{ and } \operatorname{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \leq \Delta\}|. \tag{3.5}$$

Let us show that the diagonal sum $\sum_Q m_{Q,Q}$ is negligible. First, estimate

$$\sum_{Q \in \mathcal{Q}} m_{Q,Q} \leq \sum_{Q} |\{(\omega_1, \omega_2) \in \mathbf{X}^2 : z_{\omega_1}, z_{\omega_2} \in Q\}| \leq |\mathbf{X}| \cdot \max_{Q \in \mathcal{Q}} |\{\omega \in \mathbf{X} : z_\omega \in Q\}|.$$

Writing $Q = [a_0, a_0 + \Delta) \times [b_0, b_0 + \Delta) \in \mathcal{D}_\Delta([-1,1)^2)$, it is easy to check that $\{\omega \in \mathbf{X} : z_\omega \in Q\} \subset \mathbf{R}_{\Delta \times \Delta \times 1}(a_0, b_0, 0)$. Since $\mathbf{X}$ was assumed to be a Frostman $(\delta, \alpha, \beta, \Delta^{-\eta})$-set, we may infer that $|\{\omega \in \mathbf{X} : z_\omega \in Q\}| \leq \Delta^{\alpha - \eta}|\mathbf{X}|$. In particular, since we assumed at (3.3) that $\alpha - \eta > 1 + \varepsilon > 1 + \varepsilon/2$,

$$\sum_{Q \in \mathcal{Q}} m_{Q,Q} \leq \tfrac{1}{10} \Delta^{1+\varepsilon/2} |\mathbf{X}|^2,$$

provided $\Delta > 0$ is sufficiently small. As a consequence of this and (3.4),

$$\sum_{Q_1 \neq Q_2} m_{Q_1,Q_2} \gtrsim \Delta^{1+\varepsilon/2} |\mathbf{X}|^2.$$

We now apply Lemma 3.1 to the weighted directed graph $G = (\mathcal{Q}, E, \{w(e)\}_{e \in E})$, where $E = \{(Q_1, Q_2) : Q_1 \neq Q_2\}$ and $w(Q_1, Q_2) = m_{Q_1,Q_2}$. The result is a bi-partite subgraph $(\mathcal{A} \cup \mathcal{B}, \overline{\mathcal{E}})$, where $\mathcal{A}, \mathcal{B} \subset \mathcal{Q}$ are disjoint, $\overline{\mathcal{E}} = \{(Q_1, Q_2) \in \mathcal{E} : Q_1 \in \mathcal{A} \text{ and } Q_2 \in \mathcal{B}\}$, and

$$\sum_{(Q_1,Q_2) \in \overline{\mathcal{E}}} m_{Q_1,Q_2} \geq \tfrac{1}{4} \sum_{Q_1 \neq Q_2} m_{Q_1,Q_2} \gtrsim \Delta^{1+\varepsilon/2} |\mathbf{X}|^2.$$

Define $\mathbf{A} := \{\omega \in \mathbf{X} : z_\omega \in \cup\mathcal{A}\}$ and $\mathbf{B} := \{\omega \in \mathbf{X} : z_\omega \in \cup\mathcal{B}\}$. The sets $\mathbf{A}, \mathbf{B}$ are spatially Δ-separated, since $\mathcal{A}, \mathcal{B}$ are disjoint subsets of the Δ-separated family $\mathcal{Q}$. Finally, recalling the definition of m_{Q_1,Q_2} from (3.5), and using $\overline{\mathcal{E}} \subset \mathcal{E}$,

$$\begin{aligned}
\mathcal{I}^+_\Delta(\mathbf{A}, \mathbf{B}) &\overset{\text{def.}}{=} |\{(\omega_1, \omega_2) \in \mathbf{A} \times \mathbf{B} : \operatorname{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \leq \Delta\}| \\
&= \sum_{(Q_1,Q_2) \in \mathcal{A} \times \mathcal{B}} |\{(\omega_1, \omega_2) \in \mathbf{X}^2 : z_{\omega_1} \in Q_1,\ z_{\omega_2} \in Q_2 \text{ and } \operatorname{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \leq \Delta\}| \\
&\overset{\text{def.}}{=} \sum_{(Q_1,Q_2) \in \overline{\mathcal{E}}} m_{Q_1,Q_2} \gtrsim \Delta^{1+\varepsilon/2} |\mathbf{X}|^2.
\end{aligned}$$

This proves (3.2) for $\Delta > 0$, depending on ε. □

3.2. Proof of Theorem B. Now we use Lemma 3.3 along with the *high-low inequality* (for rays) to prove Theorem B. If one wished to prove Theorem B with lines in place of rays, the version of the high-low inequality in [CPZ25, Theorem 1.7] stated in (3.9) would suffice. We give the statement of an appropriate version for rays in Theorem 3.5 below, but postpone the proof to Appendix A.

Let $\mathbf{A}, \mathbf{B} \subset \Omega$ be finite sets. The reader may think of $\mathbf{A}$ as parametrising a family of rays, and $\mathbf{B}$ as parametrising a family of points. Accordingly, define

$$g := \sum_{\omega \in \mathbf{B}} \delta_{z_\omega} \quad \text{and} \quad f_w := w^{-1} \sum_{\omega \in \mathbf{A}} \mathbf{1}_{[\ell_\omega^+]_{w/2}}, \; w \in (0,1],$$

where $[\ell_\omega^+]_{w/2}$ is the open $(w/2)$-neighbourhood of ℓ_ω^+. Let $\chi \in C_c^\infty(\mathbb{R}^2)$ be a fixed non-negative function with $\operatorname{spt} \chi \subset B(1)$. For $w > 0$, write $\chi_w := w^{-2}\chi(\cdot/w)$, and define

$$\mathcal{J}_w^+(\mathbf{A}, \mathbf{B}) := w \cdot \langle \chi_w * \chi_{w/2} * f_w, g \rangle := w \cdot \int \chi_w * \chi_{w/2} * f_w \, dg. \tag{3.6}$$

The definition of $\mathcal{J}_w^+$ depends on χ, but we suppress this from the notation. The quantity $\mathcal{J}_w^+$ should be viewed as a variant of the incidence count $\mathcal{I}_w^+$ which is more amenable to L^2-orthogonality arguments; we will see this benefit in the proof of Theorem 3.5. For now, we only need to know that $\mathcal{J}_w^+$ is comparable to $\mathcal{I}_w^+$ in the following sense:

Lemma 3.4. *If $\chi \in C_c^\infty(\mathbb{R}^2)$ is arbitrary with $\operatorname{spt} \chi \subset B(1)$, then $\mathcal{J}_w^+(\mathbf{A}, \mathbf{B}) \lesssim_\chi \mathcal{I}_{2w}^+(\mathbf{A}, \mathbf{B})$. Moreover, if $\chi \geq \mathbf{1}_{B(1/2)}$, then $\mathcal{J}_w^+(\mathbf{A}, \mathbf{B}) \gtrsim \mathcal{I}_w^+(\mathbf{A}, \mathbf{B})$.*

Proof. The upper bound follows from $\chi_w * \chi_{w/2} * \mathbf{1}_{[\ell_\omega^+]_{w/2}} \lesssim_\chi \mathbf{1}_{[\ell_\omega^+]_{2w}}$, which yields

$$\mathcal{J}_w^+(\mathbf{A}, \mathbf{B}) \lesssim_\chi \sum_{\omega_1 \in \mathbf{A}} \sum_{\omega_2 \in \mathbf{B}} \mathbf{1}_{[\ell_{\omega_1}^+]_{2w}}(z_{\omega_2}) = \mathcal{I}_{2w}^+(\mathbf{A}, \mathbf{B}),$$

as desired. If $\chi \geq \mathbf{1}_{B(1/2)}$, then we also have a nearly matching lower bound $\chi_w * \chi_{w/2} * \mathbf{1}_{[\ell_\omega^+]_{w/2}} \gtrsim \mathbf{1}_{[\ell_\omega^+]_{w/2}}$, which yields the stated lower bound. □

Before stating the high-low theorem, we need one more piece of notation: for $\mathbf{X} \subset \Omega$ and $w > 0$, define (following the notation in [CPZ25])

$$M_{w \times w}(\mathbf{X}) := \sup_{z_0 \in \mathbb{R}^2} |\{\omega \in \mathbf{X} : z_\omega \in B(z_0, w)\}| \tag{3.7}$$

and

$$M_{1 \times w}(\mathbf{X}) := \sup_{\ell_0 \in \mathcal{A}(2,1)} |\{\omega \in \mathbf{X} : \ell_\omega \in B_{d_{\mathcal{A}}}(\ell_0, w)\}|. \tag{3.8}$$

The ball $B_{d_{\mathcal{A}}}(\ell_0, w) \subset \mathcal{A}(2,1)$ is defined relative to the following metric $d_{\mathcal{A}}$ on $\mathcal{A}(2,1)$ (the same metric as in [CPZ25, (3)]). For $\ell_1, \ell_2 \in \mathcal{A}(2,1)$ with slopes $\sigma_1, \sigma_2 \in [-1,1]$, we set

$$d_{\mathcal{A}}(\ell_1, \ell_2) := |\operatorname{dist}(\ell_1, 0) - \operatorname{dist}(\ell_2, 0)| + |\sigma_1 - \sigma_2|.$$

We are then prepared to state the high-low inequality for rays:

Theorem 3.5. *Let $\mathbf{A}, \mathbf{B} \subset \Omega$ be finite sets. Then,*

$$\left| \frac{\mathcal{J}_w^+(\mathbf{A}, \mathbf{B})}{w|\mathbf{A}||\mathbf{B}|} - \frac{\mathcal{J}_{w/2}^+(\mathbf{A}, \mathbf{B})}{(w/2)|\mathbf{A}||\mathbf{B}|} \right| \leq A_\chi \left(\frac{M_{1 \times w}(\mathbf{A}) M_{w \times w}(\mathbf{B})}{|\mathbf{A}||\mathbf{B}|} \cdot w^{-3} \log w^{-1} \right)^{1/2}, \quad w \in (0, \tfrac{1}{2}],$$

where the constant $A_\chi > 0$ depends only on χ.

Remark 3.6. Let us compare Theorem 3.5 to [CPZ25, Theorem 1.7]. First of all, one could also define $\mathcal{J}_w(\mathbf{A},\mathbf{B})$ as above, but with the ray ℓ_ω^+ inside the definition of f_w replaced by the line ℓ_ω. Then the following holds:

$$\left|\frac{\mathcal{J}_w(\mathbf{A},\mathbf{B})}{w|\mathbf{A}||\mathbf{B}|} - \frac{\mathcal{J}_{w/2}(\mathbf{A},\mathbf{B})}{(w/2)|\mathbf{A}||\mathbf{B}|}\right| \leq A_\chi\Big(\frac{M_{1\times w}(\mathbf{A})M_{w\times w}(\mathbf{B})}{|\mathbf{A}||\mathbf{B}|}\cdot w^{-3}\Big)^{1/2}, \quad w \in (0,1]. \tag{3.9}$$

This is proved in [CPZ25, Appendix A.1] (and it implies the high-low inequality as stated in [CPZ25, Theorem 1.7]). The inequality (3.9) is a "$\log w^{-1}$" factor sharper than the one stated in Theorem 3.5.

We are then equipped to prove Theorem B.

Proof (of Theorem B). Let $\chi \in C_c^\infty(\mathbb{R}^2)$ be a non-negative function satisfying $\mathbf{1}_{B(1/2)} \leq \chi \leq \mathbf{1}_{B(1)}$, and let $A \geq 1$ be an absolute constant to be determined, which in particular is larger than the constant A_χ from Theorem 3.5 associated to χ. Let $\mathbf{X} \subset \Omega$ be a Frostman (δ,α,β,C)-set. Let $\eta = \eta(\alpha,\beta,\varepsilon) > 0$ and $\Delta_0 = \Delta_0(\alpha,\beta,\varepsilon) > 0$ be the constants given by Lemma 3.3 with

$$\varepsilon := \varepsilon(\alpha,\beta) := \frac{\alpha+\beta-3}{6} \in (0,1].$$

(Note that η, Δ_0 actually only depend on α,β.) Finally, choose

$$\kappa := \kappa(\alpha,\beta) := \frac{1}{\min\{\eta,\varepsilon\}}.$$

We claim that Theorem B holds with this choice of κ.

We begin by choosing an intermediate scale Δ. The first requirement is that $\Delta \leq \min\{\Delta_0, C^{-1/\eta}\}$. This guarantees that Lemma 3.3 can be applied at scale Δ (since $\mathbf{X}$ is a Frostman $(\delta,\alpha,\beta,\Delta^{-\eta})$-set). The second requirement is that Δ is sufficiently small so that the following inequality holds:

$$\Delta^\varepsilon - A\sum_{w\in 2^{-\mathbb{N}}\cap(0,\Delta]}(C^2 w^{\alpha+\beta-3}\log w^{-1})^{1/2} \geq \tfrac{1}{2}\Delta^\varepsilon. \tag{3.10}$$

This can be arranged: if $\Delta = 2^{-N}$ is sufficiently small depending only on ε, it holds $k^{1/2} \leq 2^{-\varepsilon k}$ for all $k \geq N$. Thus there is a constant $c_\varepsilon > 0$ so that if $\Delta \leq c_\varepsilon C^{-1/\varepsilon}$,

$$\sum_{k=N}^\infty k^{1/2}2^{-3\varepsilon k} \leq \sum_{k=N}^\infty 2^{-2\varepsilon k} \leq \frac{\Delta^{2\varepsilon}}{1-2^{-2\varepsilon}} \leq \frac{\Delta^\varepsilon}{2AC}.$$

Let $\Delta \in 2^{-\mathbb{N}}$ be chosen so that

$$\Delta \leq \min\{C^{-1/\eta}, c_\varepsilon C^{-1/\varepsilon}, \Delta_0\} < 2\Delta \tag{3.11}$$

Let $\mathbf{A},\mathbf{B} \subset \mathbf{X}$ be the spatially Δ-separated subsets obtained by Lemma 3.3 applied at scale Δ.

First, note that the conclusion (3.2) combined with Lemma 3.4 implies $\mathcal{J}_\Delta^+(\mathbf{A},\mathbf{B}) \gtrsim \mathcal{I}_\Delta^+(\mathbf{A},\mathbf{B}) \geq \Delta^{1+\varepsilon}|\mathbf{X}|^2$. It follows from the Frostman (δ,α,β,C)-set property of $\mathbf{X}$ that

$$M_{w\times w}(\mathbf{B}) \lesssim Cw^\alpha|\mathbf{X}| \quad \text{and} \quad M_{1\times w}(\mathbf{A}) \lesssim Cw^\beta|\mathbf{X}|, \quad w \in 2^{-\mathbb{N}}. \tag{3.12}$$

(To check this detail, note that sets of the form $\{\omega : z_\omega \in B(z_0, w)\}$ and $\{\omega : \ell_\omega \in B_{d_\mathbb{A}}(\ell_0, w)\}$ appearing in the definitions of $M_{w\times w}(\mathbf{B})$ and $M_{1\times w}(\mathbf{A})$ are contained, respectively, in phase space rectangles of the form $\mathbf{R}_{Aw\times Aw\times 1}(\omega_0)$ and $\mathbf{R}_{1\times Aw\times Aw}(\omega_0)$, where $A \geq 1$ is absolute. This is discussed in [CPZ25, (17)].)

Fix $w \in [\delta/2, \Delta]$, and apply the high-low inequality (Theorem 3.5) at scale w as follows:

$$\begin{aligned}\left|\frac{\mathcal{J}_w^+(\mathbf{A},\mathbf{B})}{w|\mathbf{X}|^2} - \frac{\mathcal{J}_{w/2}^+(\mathbf{A},\mathbf{B})}{(w/2)|\mathbf{X}|^2}\right| &= \frac{|\mathbf{A}||\mathbf{B}|}{|\mathbf{X}|^2}\left|\frac{\mathcal{J}_w^+(\mathbf{A},\mathbf{B})}{w|\mathbf{A}||\mathbf{B}|} - \frac{\mathcal{J}_{w/2}^+(\mathbf{A},\mathbf{B})}{(w/2)|\mathbf{A}||\mathbf{B}|}\right| \\ &\leq A_\chi\left(\frac{|\mathbf{A}||\mathbf{B}|}{|\mathbf{X}|^2}\right)^{1/2}\left(\frac{M_{1\times w}(\mathbf{A})M_{w\times w}(\mathbf{B})}{|\mathbf{X}|^2}\cdot w^{-3}\log w^{-1}\right)^{1/2} \\ &\overset{(3.12)}{\leq} A(C^2 w^{\alpha+\beta-3}\log w^{-1})^{1/2}.\end{aligned}$$

Finally, summing over $w \in [\delta/2, \Delta]$ and using Lemma 3.4,

$$\begin{aligned}\frac{\mathcal{I}_\delta^+(\mathbf{A},\mathbf{B})}{\delta|\mathbf{X}|^2} \gtrsim \frac{\mathcal{J}_{\delta/2}^+(\mathbf{A},\mathbf{B})}{\delta|\mathbf{X}|^2} &\geq \Delta^\varepsilon - A\sum_{w\in 2^{-\mathbb{N}}\cap(0,\Delta]}(C^2 w^{\alpha+\beta-3}\log w^{-1})^{1/2} \\ &\overset{(3.10)}{\geq} \tfrac{1}{2}\Delta^\varepsilon \overset{(3.11)}{\gtrsim_{\alpha,\beta}} C^{-\max\{\varepsilon/\eta,1\}}.\end{aligned}$$

Since $\mathbf{A}, \mathbf{B}$ are spatially Δ-separated, the proof of Theorem B is complete. □

4. INCIDENCE LOWER BOUND FOR RAYS WITH δ^ε-LOSS

We now begin the proof of Theorem C. The argument is virtually the same as the proof of [CPZ25, Theorem 1.9] (or Theorem 1.10), apart from a few technical difficulties caused by considering ray incidences instead of line incidences. Fortunately, we are able to use as black boxes some of the most complicated parts in the proof of [CPZ25, Theorem 1.9] which deal with the "combinatorics of Lipschitz functions".

4.1. An initial estimate for rays. The purpose of this section is to prove Lemma 4.5. This is a counterpart of the "initial estimate" in [CPZ25, Proposition 4.2], with $\mathcal{I}_\eta^+$ in place of $\mathcal{I}_\eta$.

Definition 4.1 (Uniformity). Let $\mathcal{S} = \{u_j \times v_j \times w_j\}_{j\in J}$ be a family of scale triples satisfying $v_j \geq u_j w_j$, and let $K \geq 1$. A finite set $\mathbf{X} \subset \Omega$ is called *K-uniform on $\mathcal{S}$* if for every $j \in J$, $\omega \in \mathbf{X}$,

$$|\mathbf{X} \cap \mathbf{R}_{u_j\times v_j\times w_j}(\omega)| \geq \frac{M_{u_j\times v_j\times w_j}(\mathbf{X})}{K}.$$

Here $M_{u\times v\times w}(\mathbf{X}) := \sup_{\omega\in\Omega}|\mathbf{X} \cap \mathbf{R}_{u\times v\times w}(\omega)|$.

We will also say that $\mathbf{X}$ *is uniform on a sequence* $\{\Delta_j\}_{j=0}^m$. This means that $\mathbf{X}$ is uniform on $\{\Delta_i \times \Delta_j \times \Delta_k : i,j,k \in \{0,\ldots,m\} \text{ and } \Delta_j \geq \Delta_i\Delta_k\}$.

Remark 4.2. In [CPZ25] the number $M_{u\times v\times w}(\mathbf{X})$ is defined as a maximum over a fixed boundedly overlapping cover of $\mathbb{R}^3$ by $(u \times v \times w)$-rectangles, see above [CPZ25, Lemma 3.2]. The two definitions of $M_{u\times v\times w}(\mathbf{X})$ are comparable up to absolute constants.

Definition 4.3 (Rectangular covering numbers). For $u, v, w \in (0,1]$ with $v \geq uw$, and $\mathbf{X} \subset \Omega$, we write $|\mathbf{X}|_{u\times v\times w}$ for the minimal number of rectangles of the form $\mathbf{R}_{u\times v\times w}(\omega)$, $\omega \in \Omega$, required to cover $\mathbf{X}$.

For K-uniform sets, there is a useful relation between $M_{u\times v\times w}(\mathbf{X})$ and $|\mathbf{X}|_{u\times v\times w}$:

Lemma 4.4. *Let $\mathbf{X} \subset \Omega$ be a finite set which is K-uniform on $\{u \times v \times w\}$. Then,*

$$M_{u\times v\times w}(\mathbf{X}) \lesssim K|\mathbf{X}|/|\mathbf{X}|_{u\times v\times w}.$$

Proof. This is [CPZ25, (27)]. □

Lemma 4.5. *Let $\eta \in 2^{-\mathbb{N}} \cap (0, \frac{1}{10}]$ and $K \geq 1$. Let $\mathbf{X} \subset \Omega$ be a finite set which is K-uniform on $\{(1 \times \eta \times \eta), (\eta \times \eta \times 1), (\eta \times \eta \times \eta)\}$. Then,*

$$\frac{\mathcal{I}^+_{10\eta}(\mathbf{X})}{\eta|\mathbf{X}|^2} \gtrsim K^{-5} \frac{|\mathbf{X}|_{\eta\times\eta\times\eta}}{\eta|\mathbf{X}|_{\eta\times\eta\times 1}|\mathbf{X}|_{1\times\eta\times\eta}}. \tag{4.1}$$

Proof. We start by disposing of a special case where $|\mathbf{X}|_{\eta\times\eta\times\eta} \leq AK^2|\mathbf{X}|_{1\times\eta\times\eta}$. Here $A \geq 1$ is an absolute constant to be determined later (above (4.8)). In this case, note that

$$\mathbf{R}_{\eta\times\eta\times 1}(\omega) \subset \{\omega' \in \Omega : |z_\omega - z_{\omega'}| \leq 3\eta\}, \qquad \omega \in \Omega,\ \eta \in (0,1]. \tag{4.2}$$

In particular, since always $z_\omega \in \ell^+_\omega$, we find

$$\mathcal{I}^+_{10\eta}(\mathbf{X}) \geq \sum_{\omega\in\mathbf{X}} |\{\omega' \in \mathbf{X} : |z_\omega - z_{\omega'}| \leq 3\eta\}| \geq \sum_{\omega\in\mathbf{X}} |\mathbf{X} \cap \mathbf{R}_{\eta\times\eta\times 1}(\omega)| \geq K^{-1}|\mathbf{X}|M_{\eta\times\eta\times 1}(\mathbf{X}),$$

using K-uniformity in the final estimate. Since further $M_{\eta\times\eta\times 1}(\mathbf{X}) \geq |\mathbf{X}|/|\mathbf{X}|_{\eta\times\eta\times 1}$, we see that $\mathcal{I}^+_{10\eta}(\mathbf{X}) \geq K^{-1}|\mathbf{X}|^2/|\mathbf{X}|_{\eta\times\eta\times 1}$. Since $|\mathbf{X}|_{\eta\times\eta\times\eta} \leq AK^2|\mathbf{X}|_{1\times\eta\times\eta}$ by assumption, this yields

$$\frac{\mathcal{I}^+_{10\eta}(\mathbf{X})}{\eta|\mathbf{X}|^2} \geq A^{-1}K^{-3} \frac{|\mathbf{X}|_{\eta\times\eta\times\eta}}{\eta|\mathbf{X}|_{\eta\times\eta\times 1}|\mathbf{X}|_{1\times\eta\times\eta}},$$

which is better than (4.1). So, in the sequel we may assume that

$$|\mathbf{X}|_{\eta\times\eta\times\eta} \geq AK^2|\mathbf{X}|_{1\times\eta\times\eta}. \tag{4.3}$$

We now construct a boundedly overlapping cover of $\mathbf{X}$ by rectangles of the form $\mathbf{R}_{2\times\eta\times\eta}(\omega)$ with $\omega \in \mathbf{X}$. Consider the map $\Pi \colon \Omega \to \mathbb{R}^2$ defined by $\Pi(a,b,\sigma) := (b - a\sigma, \sigma)$. Let $\{(a_j, b_j, \sigma_j)\}_{j\in J} \subset \mathbf{X}$ be a maximal set with the property that the points $\Pi(a_j, b_j, \sigma_j) \in \mathbb{R}^2$ are $(\eta/2)$-separated. Define $\mathcal{R} := \{\mathbf{R}_{2\times\eta\times\eta}(a_j, b_j, \sigma_j) : j \in J\}$.

We claim that $\mathbf{X} \subset \cup\mathcal{R}$. Fix $(a,b,c) \in \mathbf{X}$. By definition, there exists (a_j, b_j, σ_j), $j \in J$, such that $|(b - a\sigma) - (b_j - a_j\sigma_j)| \leq \eta/2$ and $|\sigma - \sigma_j| \leq \eta/2$. We claim that $(a, b, \sigma) \in \mathbf{R}_{2\times\eta\times\eta}(a_j, b_j, \sigma_j)$. To see this, first expand

$$(a,b,\sigma) = (a_j + (a - a_j), b_j + (b - b_j), \sigma_j + (\sigma - \sigma_j)) =: (a_j + r_1, b_j + (b - b_j), \sigma_j + r_3),$$

where $|r_1| = |a - a_j| \leq 2$ and $|r_3| \leq \eta/2$. Next, note that for some $\xi \in \mathbb{R}$ with $|\xi| \leq \eta/2$,

$$b - b_j = a\sigma - a_j\sigma_j + \xi = \sigma_j(a - a_j) + a(\sigma - \sigma_j) + \xi = \sigma_j r_1 + r_2,$$

where $|r_2| = |a(\sigma - \sigma_j) + \xi| \leq \eta$. We have managed to write (a, b, σ) in the form $(a, b, \sigma) = (a_j + r_1, b_j + \sigma_j r_1 + r_2, \sigma_j + r_3)$ for some $(r_1, r_2, r_3) \in [-2, 2] \times [-\eta, \eta]^2$, and thus $(a, b, \sigma) \in \mathbf{R}_{2\times\eta\times\eta}(a_j, b_j, \sigma_j)$.

Claim I. *The rectangles $2\mathcal{R} := \{\mathbf{R}_{2\times 2\eta\times 2\eta}(a_j, b_j, \sigma_j)\}$ have bounded overlap.*

Proof. It suffices to show that if $(x, y, z) \in \mathbf{R}_{2\times 2\eta\times 2\eta}(a, b, \sigma) \cap \mathbf{R}_{2\times 2\eta\times 2\eta}(a', b', \sigma')$, then $|\Pi(a, b, \sigma) - \Pi(a', b', \sigma')| \lesssim \eta$. Write

$$(x, y, z) = (a + r_1, b + r_1\sigma + r_2, \sigma + r_3) = (a' + r_1', b' + r_1'\sigma' + r_2', \sigma' + r_3'), \tag{4.4}$$

where $r_1, r_1' \in [-2, 2]$ and $r_2, r_2', r_3, r_3' \in [-2\eta, 2\eta]$. Then $a + r_1 = a' + r_1'$ and $|\sigma - \sigma'| \leq |r_3| + |r_3'| \leq 4\eta$. To simplify the calculation, let us pretend that $\sigma = \sigma'$; this is correct up to $O(\eta)$-terms. From (4.4) we see that

$$b - b' = (r_1' - r_1)\sigma + O(\eta) = (a - a')\sigma + O(\eta).$$

Therefore $|(b - a\sigma) - (b' - a'\sigma)| \lesssim \eta$, and finally $|\Pi(a, b, \sigma) - \Pi(a', b', \sigma')| \lesssim \eta$. □

Note that $|\mathcal{R}| \geq |\mathbf{X}|_{1\times\eta\times\eta}$ by the definition of $|\cdot|_{1\times\eta\times\eta}$, and $|\mathbf{X} \cap \mathbf{R}| \geq K^{-1} M_{1\times\eta\times\eta}(\mathbf{X})$ by K-uniformity for all $\mathbf{R} \in \mathcal{R}$ (the rectangles in $\mathcal{R}$ are centred at $\mathbf{X}$). Consequently,

$$M_{\eta\times\eta\times\eta}(\mathbf{X})|\mathbf{X} \cap \mathbf{R}|_{\eta\times\eta\times\eta} \geq |\mathbf{X} \cap \mathbf{R}| \geq K^{-1} M_{1\times\eta\times\eta}(\mathbf{X}) \geq K^{-1}|\mathbf{X}|/|\mathbf{X}|_{1\times\eta\times\eta}.$$

Rearranging, and using Lemma 4.4 to estimate

$$|\mathbf{X}| \gtrsim K^{-1} M_{\eta\times\eta\times\eta}(\mathbf{X})|\mathbf{X}|_{\eta\times\eta\times\eta}, \tag{4.5}$$

we find

$$|\mathbf{X} \cap \mathbf{R}|_{\eta\times\eta\times\eta} \gtrsim K^{-2}|\mathbf{X}|_{\eta\times\eta\times\eta}/|\mathbf{X}|_{1\times\eta\times\eta} \overset{(4.3)}{\gtrsim} A, \qquad \mathbf{R} \in \mathcal{R}. \tag{4.6}$$

Next, fix $\mathbf{R} = \mathbf{R}_{2\times\eta\times\eta}(a_0, b_0, \sigma_0) \in \mathcal{R}$, thus $(a_0, b_0, \sigma_0) \in \mathbf{X}$, and

$$\mathbf{R} = \{(a_0 + r_1, b_0 + \sigma_0 r_1 + r_2, \sigma_0 + r_3) : (r_1, r_2, r_3) \in [-2, 2] \times [-\eta, \eta]^2\}.$$

Let $P_{\mathbf{R}} \subset P[\mathbf{X} \cap \mathbf{R}] \subset [-1, 1]^2$ be a maximal η-separated set. We claim that

$$|P_{\mathbf{R}}| \gtrsim |\mathbf{X} \cap \mathbf{R}|_{\eta\times\eta\times\eta} \gtrsim A. \tag{4.7}$$

To see this, it suffices to show that $\{\omega_2 \in \mathbf{R} : |z_{\omega_2} - z_{\omega_1}| \leq \eta\} \subset B(\omega_1, 3\eta)$ for all $\omega_1 \in \mathbf{R}$. This is true, since every pair $\omega_1, \omega_2 \in \mathbf{R}$ differs in the 3^{rd} coordinate by $\leq 2\eta$.

Write $\ell_{\mathbf{R}} := \ell_{(a_0, b_0, \sigma_0)} = (a_0, b_0) + \mathbb{R}(1, \sigma_0)$. From the definition of $\mathbf{R}$, one sees that $P[\mathbf{R}] \subset [\ell_{\mathbf{R}}]_\eta$ (the η-neighbourhood of $\ell_{\mathbf{R}}$). Summarising the previous observations, $P_{\mathbf{R}}$ is an η-separated subset of $[\ell_{\mathbf{R}}]_\eta$ with cardinality $|P_{\mathbf{R}}| \gtrsim |\mathbf{X} \cap \mathbf{R}|_{\eta\times\eta\times\eta} \gtrsim A$. Thus choosing $A \geq 1$ sufficiently large, $|P_{\mathbf{R}}| \geq 2$. Since $|\sigma_0| \leq 1$, we now extract disjoint subsets $A_{\mathbf{R}}, B_{\mathbf{R}} \subset P_{\mathbf{R}}$ with cardinalities

$$|A_{\mathbf{R}}| \sim |P_{\mathbf{R}}| \sim |B_{\mathbf{R}}| \tag{4.8}$$

such that $\max\{x : (x,y) \in A_{\mathbf{R}}\} \leq \min\{x : (x,y) \in B_{\mathbf{R}}\}$. It is then easy to check that the following holds:

Claim II. *Assume that $z_1, z_2 \in \mathbb{R}^2$ and $\sigma \in \mathbb{R}$ satisfy*

$$\operatorname{dist}(z_1, A_{\mathbf{R}}) \leq 2\eta, \quad \operatorname{dist}(z_2, B_{\mathbf{R}}) \leq 2\eta, \quad \textit{and} \quad |\sigma - \sigma_0| \leq 2\eta.$$

Then $\operatorname{dist}(z_2, \ell^+_{z_1,\sigma}) \leq 10\eta$.

We now define

$$\mathbf{A}(\mathbf{R}) := \{\omega \in \mathbf{X} \cap 2\mathbf{R} : \operatorname{dist}(z_\omega, A_{\mathbf{R}}) \leq 2\eta\},$$

where $2\mathbf{R} := \mathbf{R}_{2\times 2\eta \times 2\eta}(a_0, b_0, \sigma_0)$, and

$$\mathbf{B}(\mathbf{R}) := \{\omega \in \mathbf{X} : \operatorname{dist}(z_\omega, B_{\mathbf{R}}) \leq 2\eta\}.$$

Note that if $\omega_1 = (z_1, \sigma_1) \in \mathbf{A}(\mathbf{R})$ and $\omega_2 = (z_2, \sigma_2) \in \mathbf{B}(\mathbf{R})$, then the points z_j and the slope σ_1 satisfy the hypotheses of Claim II. Thus,

$$\mathbf{A}(\mathbf{R}) \times \mathbf{B}(\mathbf{R}) \subset \{(\omega_1, \omega_2) \in \mathbf{X}^2 : \operatorname{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \leq 10\eta\}.$$

Since the rectangles $2\mathbf{R}$ have bounded overlap, this implies

$$\mathcal{I}^+_{10\eta}(\mathbf{X}) = |\{(\omega_1, \omega_2) \in \mathbf{X}^2 : \operatorname{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \leq 10\eta\}| \gtrsim \sum_{\mathbf{R} \in \mathcal{R}} |\mathbf{A}(\mathbf{R})||\mathbf{B}(\mathbf{R})|.$$

We finally claim that

$$|\mathbf{A}(\mathbf{R})| \gtrsim K^{-3} \frac{|\mathbf{X}|}{|\mathbf{X}|_{1\times\eta\times\eta}} \quad \text{and} \quad |\mathbf{B}(\mathbf{R})| \gtrsim K^{-1}|P_{\mathbf{R}}| \cdot \frac{|\mathbf{X}|}{|\mathbf{X}|_{\eta\times\eta\times 1}} \tag{4.9}$$

for all $\mathbf{R} \in \mathcal{R}$. Once these lower bound have been established, we may complete the proof as follows:

$$\begin{aligned}
\sum_{\mathbf{R} \in \mathcal{R}} |\mathbf{A}(\mathbf{R})||\mathbf{B}(\mathbf{R})| &\gtrsim K^{-4} \frac{|\mathbf{X}|^2}{|\mathbf{X}|_{1\times\eta\times\eta}|\mathbf{X}|_{\eta\times\eta\times 1}} \sum_{\mathbf{R} \in \mathcal{R}} |P_{\mathbf{R}}| \\
&\overset{(4.7)}{\gtrsim} K^{-4} \frac{|\mathbf{X}|^2}{|\mathbf{X}|_{1\times\eta\times\eta}|\mathbf{X}|_{\eta\times\eta\times 1}} \sum_{\mathbf{R} \in \mathcal{R}} |\mathbf{X} \cap \mathbf{R}|_{\eta\times\eta\times\eta} \\
&\overset{(*)}{\gtrsim} K^{-5} \frac{|\mathbf{X}|^2 |\mathbf{X}|_{\eta\times\eta\times\eta}}{|\mathbf{X}|_{1\times\eta\times\eta}|\mathbf{X}|_{\eta\times\eta\times 1}},
\end{aligned}$$

where the estimate $(*)$ follows from K-uniformity:

$$\sum_{\mathbf{R} \in \mathcal{R}} |\mathbf{X} \cap \mathbf{R}|_{\eta\times\eta\times\eta} \geq \sum_{\mathbf{R} \in \mathcal{R}} \frac{|\mathbf{X} \cap \mathbf{R}|}{M_{\eta\times\eta\times\eta}(\mathbf{X})} \geq \frac{|\mathbf{X}|}{M_{\eta\times\eta\times\eta}(\mathbf{X})} \overset{(4.5)}{\gtrsim} K^{-1}|\mathbf{X}|_{\eta\times\eta\times\eta}.$$

Let us finally prove (4.9), starting with the estimate for $|\mathbf{A}(\mathbf{R})|$. First, write

$$|\mathbf{A}(\mathbf{R})| \gtrsim \sum_{z \in A_{\mathbf{R}}} |\{\omega \in \mathbf{X} \cap 2\mathbf{R} : |z_\omega - z| \leq 2\eta\}|.$$

Fix $z \in A_{\mathbf{R}}$, and recall that $z = z_{\omega_0}$ for some $\omega_0 \in \mathbf{X} \cap \mathbf{R}$. Now $\mathbf{X} \cap \mathbf{R}_{\eta \times \eta \times \eta}(\omega_0) \subset \{\omega \in \mathbf{X} \cap 2\mathbf{R} : |z_\omega - z| \leq 2\eta\}$, so

$$|\{\omega \in \mathbf{X} \cap 2\mathbf{R} : |z_\omega - z| \leq 2\eta\}| \geq |\mathbf{X} \cap \mathbf{R}_{\eta \times \eta \times \eta}(\omega_0)| \gtrsim K^{-1} \frac{|\mathbf{X}|}{|\mathbf{X}|_{\eta \times \eta \times \eta}}$$

by K-uniformity. Recalling from (4.8) that $|A_{\mathbf{R}}| \sim |P_{\mathbf{R}}| \gtrsim |\mathbf{X} \cap \mathbf{R}|_{\eta \times \eta \times \eta}$, we get

$$|\mathbf{A}(\mathbf{R})| \gtrsim K^{-1} \frac{|\mathbf{X} \cap \mathbf{R}|_{\eta \times \eta \times \eta} |\mathbf{X}|}{|\mathbf{X}|_{\eta \times \eta \times \eta}} \overset{(4.6)}{\gtrsim} K^{-3} \frac{|\mathbf{X}|}{|\mathbf{X}|_{1 \times \eta \times \eta}},$$

as desired. The lower bound for $|\mathbf{B}(\mathbf{R})|$ is more straightforward, and is based on the following observation. Fix $z = z_{\omega_0} \in B_{\mathbf{R}}$ with $\omega_0 \in \mathbf{X}$. Then,

$$\mathbf{X} \cap \mathbf{R}_{\eta \times \eta \times 1}(\omega_0) \subset \{\omega \in \mathbf{X} : |z - z_\omega| \leq 2\eta\},$$

so $|\{\omega \in \mathbf{X} : |z - z_\omega| \leq 2\eta\}| \gtrsim K^{-1}|\mathbf{X}|/|\mathbf{X}|_{\eta \times \eta \times 1}$ by K-uniformity. Consequently,

$$|\mathbf{B}(\mathbf{R})| \gtrsim \sum_{z \in B_{\mathbf{R}}} |\{\omega \in \mathbf{X} : |z - z_\omega| \leq 2\eta\}| \gtrsim K^{-1}|P_{\mathbf{R}}| \cdot \frac{|\mathbf{X}|}{|\mathbf{X}|_{\eta \times \eta \times 1}},$$

as desired. This completes the proof of (4.9), and therefore the proof of Lemma 4.5. □

4.2. Large uniform subsets, branching functions, and effective triples. In this section we recap concepts and basic results from [CPZ25, Section 3], starting with [CPZ25, Lemma 3.6]:

Lemma 4.6. *Let $m, T \in \mathbb{N}$ and $\delta := 2^{-mT}$. Write*

$$K = K(m, T) := (Am^3 \log(1/\delta))^{m^3+2},$$

where $A \geq 1$ is a sufficiently large absolute constant. For every finite set $\mathbf{X} \subset \Omega$ there exists a subset $\mathbf{X}' \subset \mathbf{X}$ with $|\mathbf{X}'| \geq K^{-1}|\mathbf{X}|$ such that $\mathbf{X}'$ is K-uniform on the sequence $\{2^{-jT}\}_{j=0}^m$.

Definition 4.7 ((m, T)-uniformity). Let $m, T \in \mathbb{N}$. A set $\mathbf{X} \subset \Omega$ is called *(m, T)-uniform* if $\mathbf{X}$ is K-uniform on $\{2^{-jT}\}_{j=0}^m$ with constant $K = (Am^3 \log(1/\delta))^{m^3+2}$, where $\delta := 2^{-mT}$, and $A \geq 1$ is the absolute constant from Lemma 4.6.

From now on, if $m, T \in \mathbb{N}$ are fixed, the symbol "K" will refer to the constant in Definition 4.7. A crucial point is that if $m \in \mathbb{N}$ is fixed, and $\delta = 2^{-mT}$, then $K = \delta^{-o_{T \to \infty}(1)}$.

Definition 4.8 (Branching function). Let $m, T \in \mathbb{N}$, and assume that $\mathbf{X} \subset \Omega$ is K-uniform on the sequence $\{2^{-jT}\}_{j=0}^m$. Write

$$\mathbf{D}_m := \{(x, y, z) \in \tfrac{1}{m}\mathbb{Z}^3 : x, y \in [0, 1] \text{ and } z \in [0, \min\{1, x + y\}]\}.$$

We define the *branching function* $f_{\mathbf{X}}\colon \mathbf{D}_m \to [0,\infty)$ *of* $\mathbf{X}$ via the relation

$$|\mathbf{X}|_{\delta^x\times\delta^z\times\delta^y} = \delta^{-f_{\mathbf{X}}(x,y,z)}, \qquad (x,y,z)\in\mathbf{D}_m.$$

Remark 4.9. The well-posedness of $f_{\mathbf{X}}$ does not require the uniformity of $\mathbf{X}$. However, note that if $\mathbf{X}$ is uniform, and $(x,y,z) = (i/m, j/m, k/m) \in \mathbf{D}_m$, then $\delta^x \times \delta^z \times \delta^y = 2^{-iT} \times 2^{-kT} \times 2^{-jT}$ is one of the scales on which $\mathbf{X}$ is uniform.

Definition 4.10 (Renormalised branching function). Let $\mathbf{X}, \mathbf{D}_m, f_{\mathbf{X}}$ be given as in Definition 4.8. Fix $(x_0,y_0)\in\frac{1}{m}\mathbb{Z}$ such that $(x_0,y_0,x_0+y_0)\in\mathbf{D}_m$. Define

$$f_{\mathbf{X}}(x,y,z;x_0,y_0) := f_{\mathbf{X}}(x_0+x, y_0+y, x_0+y_0+z) - f_{\mathbf{X}}(x_0,y_0,x_0+y_0),$$

for all triples $(x,y,z)\in\frac{1}{m}\mathbb{Z}^3$ such that the right hand side is well-defined.

As the name suggests, $f_{\mathbf{X}}(\cdots;x_0,y_0)$ is (roughly speaking) the branching function of the renormalised restriction of $\mathbf{X}$ to a rectangle of the form $\mathbf{R}_{\delta^{x_0}\times\delta^{x_0+y_0}\times\delta^{y_0}}(\omega)$. Lemma 4.12 (which is [CPZ25, Lemma 3.11]) will make this precise.

Definition 4.11 (Rescaling map). Let $u,w\in(0,1]$ and $\omega_0 = (a_0,b_0,\sigma_0)\in\Omega$. For $\mathbf{R} := \mathbf{R}_{u\times uw\times w}(\omega_0)$, we define the *rescaling map*

$$\psi_{\mathbf{R}}(\omega) := \Big(\frac{a-a_0}{u}, \frac{b-(b_0+\sigma_0(a-a_0))}{uw}, \frac{\sigma-\sigma_0}{w}\Big), \qquad \omega=(a,b,\sigma)\in\Omega.$$

Lemma 4.12. *Let $m,T\in\mathbb{N}$, and let $\mathbf{X}\subset\Omega$ be (m,T)-uniform. Let $(x_0,y_0,z_0)\in\mathbf{D}_m$, $\omega_0\in\mathbf{X}$, and let $\mathbf{R} := \mathbf{R}_{\delta^{x_0}\times\delta^{x_0+y_0}\times\delta^{y_0}}(\omega_0)$. Finally, let $\mathbf{X}'\subset\psi_{\mathbf{R}}(\mathbf{X}\cap\mathbf{R})$ be an (m',T)-uniform subset with $|\mathbf{X}'|\geq K^{-1}|\mathbf{X}\cap\mathbf{R}|$, where $m' := m(1-x_0-y_0)$. Then,*

$$|\mathbf{X}'|_{\delta^x\times\delta^z\times\delta^y} \sim_{K^7} \frac{|\mathbf{X}|_{\delta^{x_0+x}\times\delta^{x_0+y_0+z}\times\delta^{y_0+y}}}{|\mathbf{X}|_{\delta^{x_0}\times\delta^{x_0+y_0}\times\delta^{y_0}}} = \delta^{-f_{\mathbf{X}}(x,y,z;x_0,y_0)}, \qquad (x,y,z)\in\mathbf{D}_{m'}.$$

Definition 4.13 (Effective triple for X). Let $m,T\in\mathbb{N}$, and let $\mathbf{X}\subset\Omega$ be (m,T)-uniform with branching function $f_{\mathbf{X}}\colon\mathbf{D}_m\to[0,\infty)$. Define the functions auxiliary functions

$$b_{\mathbf{X}}(t;x,y) := f_{\mathbf{X}}(t,t,t;x,y) - (f_{\mathbf{X}}(t,0,t;x;y) + f_{\mathbf{X}}(0,t,t;x,y) - t) \tag{4.10}$$

and

$$e_{\mathbf{X}}(s;x,y) := \tfrac{1}{2}(f_{\mathbf{X}}(s,0,s;x,y) + f_{\mathbf{X}}(0,s,s;x,y) - 3s), \tag{4.11}$$

for all parameters $x,y,s,t\in\frac{1}{m}\mathbb{Z}$ such that the right hand sides are well-defined.

Let $(x,y)\in\frac{1}{m}\mathbb{Z}^2$ and $t\in\frac{1}{m}\mathbb{Z}$ with $0\leq t\leq 1-(x+y)$, and let $c_1,c_2\geq 0$. A triple $(t;x,y)$ is called (c_1,c_2)-*effective for* $\mathbf{X}$ if $\max\{t,x,y\}\leq c_2\leq\frac{1}{3}$, and

$$b_{\mathbf{X}}(t;x,y) + e_{\mathbf{X}}(s;x,y)\geq c_1, \qquad s\in[t,1-(x+y)].$$

Remark 4.14. The meaning of the functions $b_{\mathbf{X}}, e_{\mathbf{X}}$ is explained well in [CPZ25, Section 4.2], but let us say a few words. If $x=0=y$, then by definition

$$\delta^{-b_{\mathbf{X}}(t;0,0)} = \frac{|\mathbf{X}|_{\delta^t\times\delta^t\times\delta^t}}{\delta^t|\mathbf{X}|_{\delta^t\times\delta^t\times 1}|\mathbf{X}|_{1\times\delta^t\times\delta^t}}.$$

This is the quantity appearing on the right hand side of the "initial estimate", Lemma 4.5 with $\eta = \delta^t$. So, Lemma 4.5 tells us roughly speaking that $\mathcal{I}^+_{\delta^t}(\mathbf{X})/(\delta^t|\mathbf{X}|^2) \geq \delta^{-\beta_{\mathbf{X}}(t;0,0)}$.

On the other hand, the function $e_{\mathbf{X}}$ could be called the "high-low error", because

$$\delta^{e_{\mathbf{X}}(s;0,0)} = \left(|\mathbf{X}|^{-1}_{\delta^s\times\delta^s\times 1}|\mathbf{X}|^{-1}_{1\times\delta^s\times\delta^s}\delta^{-3s}\right)^{1/2}.$$

For uniform sets $\mathbf{X}$, the right hand side roughly matches the right hand side of the high-low inequality (Theorem 3.5) with $\mathbf{A} = \mathbf{X} = \mathbf{B}$; we will systematically ignore the eventually harmless $\log w^{-1}$ factor (and various other constants) in this heuristic discussion.

Now, assume that $(t;0,0)$ happens to be a (c_1,c_2)-effective triple. Then, combining the "initial estimate" (Lemma 4.5) and the high-low inequality,

$$\begin{aligned}\frac{\mathcal{I}^+_{\delta}(\mathbf{X})}{\delta|\mathbf{X}|^2} &\geq \frac{\mathcal{I}^+_{\delta^t}(\mathbf{X})}{\delta^t|\mathbf{X}|^2} - \sum_{s\in[t,1]}\left(|\mathbf{X}|^{-1}_{\delta^s\times\delta^s\times 1}|\mathbf{X}|^{-1}_{1\times\delta^s\times\delta^s}\delta^{-3s}\right)^{1/2}\\ &\geq \delta^{-b_{\mathbf{X}}(t;0,0)}\left(1 - \sup_{s\in[t,1]}\delta^{b_{\mathbf{X}}(t;0,0)+e_{\mathbf{X}}(s;0,0)}\right) \geq \delta^{-b_{\mathbf{X}}(t;0,0)}(1-\delta^{c_1}).\end{aligned}$$

Provided $\delta^{c_1} \leq \frac{1}{2}$, this shows that $\mathcal{I}^+_{\delta}(\mathbf{X})/(\delta|\mathbf{X}|^2) \geq \frac{1}{2}\delta^{-b_{\mathbf{X}}(t;0,0)}$. Moreover, $\delta^{-b_{\mathbf{X}}(t;0,0)} \geq \delta^{O(c_2)}$ thanks to the (effective triple) assumption $t \leq c_2$, and the Lipschitz continuity of $t \mapsto b_{\mathbf{X}}(t;0,0)$. In summary, a (c_1,c_2)-effective triple of the form $(t;0,0)$ yields $\mathcal{I}^+_{\delta}(\mathbf{X})/(\delta|\mathbf{X}|^2) \geq \delta^{O(c_2)}$. The same conclusion remains true for all (c_1,c_2)-effective triples $(t;x,y)$, roughly speaking by applying the argument above to $\mathbf{X} \cap \mathbf{R}$, where $\mathbf{R}$ is a $(\delta^x \times \delta^{x+y} \times \delta^y)$-rectangle. For the details, see Proposition 4.16.

For the proof of Proposition 4.16, we record a corollary of the high-low theorem for rays (Theorem 3.5) for uniform sets. This is a counterpart of [CPZ25, Proposition 4.1].

Corollary 4.15. *Let $w \in 2^{-\mathbb{N}} \cap (0,\frac{1}{2}]$, $K \geq 1$, and let $\mathbf{X} \subset \Omega$ be a finite set which is K-uniform on the scales $\{w \times w \times 1, 1 \times w \times w\}$. Then, for every $A \in 2^{\mathbb{N}}$ with $Aw \leq \frac{1}{2}$, it holds*

$$\left|\frac{\mathcal{J}^+_w(\mathbf{X})}{w|\mathbf{X}|^2} - \frac{\mathcal{J}^+_{w/A}(\mathbf{X})}{(w/A)|\mathbf{X}|^2}\right| \lesssim KA^2\left(|\mathbf{X}|^{-1}_{w\times w\times 1}|\mathbf{X}|^{-1}_{1\times w\times w}w^{-3}\log w^{-1}\right)^{1/2}, \tag{4.12}$$

and

$$\left|\frac{\mathcal{J}^+_w(\mathbf{X})}{w|\mathbf{X}|^2} - \frac{\mathcal{J}^+_{Aw}(\mathbf{X})}{(Aw)|\mathbf{X}|^2}\right| \lesssim KA^{1/2}\left(|\mathbf{X}|^{-1}_{w\times w\times 1}|\mathbf{X}|^{-1}_{1\times w\times w}w^{-3}\log w^{-1}\right)^{1/2}. \tag{4.13}$$

The implicit constants depend on the function "χ" for which $\mathcal{J}^+$ has been defined, see (3.6).

Proof. Fix $2^j \in 2^{\mathbb{N}} \cap [1, A/2]$. Applying Theorem 3.5 with $\mathbf{A} = \mathbf{X} = \mathbf{B}$, and using the trivial estimates $M_{2^{-j}w\times 2^{-j}w}(\mathbf{X}) \leq M_{w\times w}(\mathbf{X})$ and $M_{1\times 2^{-j}w}(\mathbf{X}) \leq M_{1\times w}(\mathbf{X})$,

$$\left|\frac{\mathcal{J}^+_{2^{-j}w}(\mathbf{X})}{(2^{-j}w)|\mathbf{X}|^2} - \frac{\mathcal{J}^+_{2^{-j-1}w}(\mathbf{X})}{(2^{-j-1}w)|\mathbf{X}|^2}\right| \lesssim \left(\frac{M_{w\times w}(\mathbf{X})M_{1\times w}(\mathbf{X})}{|\mathbf{X}|^2}(2^{-j}w)^{-3}\log(2^{-j}w)^{-1}\right)^{1/2}. \tag{4.14}$$

To proceed, note that $M_{1\times w}(\mathbf{X})/|\mathbf{X}| \lesssim K|\mathbf{X}|^{-1}_{w\times w\times 1}$ and $M_{1\times w}(\mathbf{X})/|\mathbf{X}| \lesssim K|\mathbf{X}|^{-1}_{1\times w\times w}$ by K-uniformity, see [CPZ25, (27)]. We mention that $M_{w\times w}(\mathbf{X})$ and $M_{1\times w}(\mathbf{X})$ defined at

(3.7)-(3.8) are *a priori* different from $M_{w\times w\times 1}(\mathbf{X})$ and $M_{1\times w\times w}(\mathbf{X})$ appearing in [CPZ25, (27)]. However, $M_{w\times w}(\mathbf{X}) \sim M_{w\times w\times 1}(\mathbf{X})$ and $M_{1\times w}(\mathbf{X}) \sim M_{1\times w\times w}(\mathbf{X})$ by [CPZ25, (21)].

Note also the general inequality $(ab)^3\log(ab) \leq a^4b^3\log b$, valid for $a \geq 1$ and $b \geq 2$ (in particular $a = 2^j$ and $b = w^{-1}$). Substituting these estimates back into (4.14),

$$\left|\frac{\mathcal{J}^+_{2^{-j}w}(\mathbf{X})}{(2^{-j}w)|\mathbf{X}|^2} - \frac{\mathcal{J}^+_{2^{-j-1}w}(\mathbf{X})}{(2^{-j-1}w)|\mathbf{X}|^2}\right| \lesssim 2^{2j}K\left(|\mathbf{X}|^{-1}_{w\times w\times 1}|\mathbf{X}|^{-1}_{1\times w\times w}w^{-3}\log w^{-1}\right)^{1/2}.$$

Summing this estimate over $2^j \in [1, A]$ yields (4.12).

For (4.13), we also fix $2^j \in 2^{\mathbb{N}} \cap [1, A/2]$, and start with

$$\left|\frac{\mathcal{J}^+_{2^jw}(\mathbf{X})}{(2^jw)|\mathbf{X}|^2} - \frac{\mathcal{J}^+_{2^{j+1}w}(\mathbf{X})}{(2^{j+1}w)|\mathbf{X}|^2}\right| \lesssim \left(\frac{M_{2^{j+1}w\times 2^{j+1}w}(\mathbf{X})M_{1\times 2^{j+1}w}(\mathbf{X})}{|\mathbf{X}|^2}(2^jw)^{-3}\log(2^jw)^{-1}\right)^{1/2}.$$

Now we estimate $M_{2^{j+1}w\times 2^{j+1}w}(\mathbf{X}) \lesssim 2^{2j}M_{w\times w}(\mathbf{X})$ and $M_{1\times 2^{j+1}w}(\mathbf{X}) \lesssim 2^{2j}M_{1\times w}(\mathbf{X})$, and $(2^jw)^{-3}\log(2^jw)^{-1} \leq 2^{-3j}w^{-3}\log w^{-1}$ to get

$$\left|\frac{\mathcal{J}^+_{2^jw}(\mathbf{X})}{(2^jw)|\mathbf{X}|^2} - \frac{\mathcal{J}^+_{2^{j+1}w}(\mathbf{X})}{(2^{j+1}w)|\mathbf{X}|^2}\right| \lesssim 2^{j/2}K\left(|\mathbf{X}|^{-1}_{w\times w\times 1}|\mathbf{X}|^{-1}_{1\times w\times w}w^{-3}\log w^{-1}\right)^{1/2}.$$

Finally, (4.13) follows by summing over $2^j \in 2^{\mathbb{N}} \cap [1, A/2]$. □

4.3. Effective triples yield ray incidences. Proposition 4.16 below is our counterpart of [CPZ25, Proposition 4.2] where line incidences have been replaced by ray incidences.

Proposition 4.16. *Let $c_1, c_2 > 0$. Then there exist $m_0 = m_0(c_1) \in \mathbb{N}$ such that the following holds for all $m \geq m_0$ and $T \geq T_0(c_1, c_2, m)$:*

Write $\delta := 2^{-mT}$. Let $\mathbf{X} \subset \Omega$ be a finite (m, T)-uniform set. Assume that there exists a triple $(t; x, y)$ which is (c_1, c_2)-effective for $\mathbf{X}$. Then,

$$\mathcal{I}^+_{10\delta}(\mathbf{X}) \geq \delta^{1+33c_2}|\mathbf{X}|^2. \tag{4.15}$$

Proof. The argument is a rigorous version of the sketch provided in Remark 4.14. We start by fixing parameters. First, let $m_0 = m_0(c_1) \in \mathbb{N}$ be so large that

$$3/m_0 < c_1/2. \tag{4.16}$$

Fix $m \geq m_0$, and let $a, c > 0$ and $C \geq 1$ be absolute constants to be determined later. Then, let $T_0 = T_0(m, c_1, c_2) \in \mathbb{N}$ be so large that the following holds:

$$\max\{2CK^{30}a^{-2}\delta^{c_1/2}(\log\tfrac{1}{\delta})^2, 2K^{21}\delta^{c_2}\} \leq c, \qquad T \geq T_0. \tag{4.17}$$

This is possible, since $K = \delta^{-o_{T\to\infty}(1)}$ for $m \in \mathbb{N}$ fixed, recalling that $\delta = 2^{-mT}$. There will be additional requirements on the size of T, depending on m, c_1, c_2, to be described later.

We claim that we may assume, without loss of generality, that

$$|\mathbf{X}|_{\delta\times\delta\times 1} \geq \delta^{-1-32c_2}. \tag{4.18}$$

The point is that (4.15) is easy to show in the opposite case. Assume that (4.18) fails. Let $b > 0$ be an absolute constant to be determined in a moment. Let $P_\delta \subset P[\mathbf{X}]$ be a maximal 10δ-separated subset, so $|P_\delta| \sim |\mathbf{X}|_{\delta\times\delta\times 1}$. Then, note that

$$\mathcal{I}^+_{10\delta}(\mathbf{X}) \geq \sum_{z\in P_\delta} |\{(\omega_1,\omega_2) \in \mathbf{X}^2 : \max\{|z_{\omega_1} - z|, |z_{\omega_2} - z| < 5\delta\}\}|,$$

since the sets on the right are disjoint, and each pair (ω_1,ω_2) in any of these sets satisfies $\operatorname{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \leq 10\delta$. Next, note that for $z = z_{\omega_0} \in P_\delta$ fixed with $\omega_0 \in \mathbf{X}$,

$$\begin{aligned}|\{(\omega_1,\omega_2) \in \mathbf{X}^2 : \max\{|z_{\omega_1} - z|, |z_{\omega_2} - z| < 5\delta\}| &= |\{\omega \in \mathbf{X} : |z_\omega - z| < 5\delta\}|^2 \\ &\overset{(4.2)}{\geq} |\mathbf{X} \cap \mathbf{R}_{\delta\times\delta\times 1}(\omega_0)|^2 \\ &\geq K^{-2} M_{\delta\times\delta\times 1}(\mathbf{X})^2\end{aligned}$$

by K-uniformity. Thus, using also $|\mathbf{X}| \leq M_{\delta\times\delta\times 1}(\mathbf{X})|\mathbf{X}|_{\delta\times\delta\times 1} \leq M_{\delta\times\delta\times 1}(\mathbf{X})\delta^{-1-32c_2}$ (by hypothesis), we obtain

$$\mathcal{I}^+_{10\delta}(\mathbf{X}) \gtrsim K^{-2}|\mathbf{X}|_{\delta\times\delta\times 1} M_{\delta\times\delta\times 1}(\mathbf{X})^2 \geq K^{-2}|\mathbf{X}|M_{\delta\times\delta\times 1}(\mathbf{X}) \geq K^{-2}\delta^{1+32c_2}|\mathbf{X}|^2.$$

This proves (4.15) for $T \in \mathbb{N}$ large enough (depending on c_2). In the sequel we may therefore assume (4.18).

Fix $\omega_0 \in \mathbf{X}$ arbitrarily, and let $\mathbf{R} := \mathbf{R}_{\delta^x\times\delta^{x+y}\times\delta^y}(\omega_0)$. As in Lemma 4.12, let $\mathbf{X}' \subset \psi_{\mathbf{R}}(\mathbf{X}\cap\mathbf{R})$ be an (m',T)-uniform subset with $|\mathbf{X}'| \geq K^{-1}|\mathbf{X}\cap\mathbf{R}|$, where $m' = m(1-x-y)$ (this subset is provided by Lemma 4.6). By Lemma 4.12,

$$|\mathbf{X}'|_{\delta^{x'}\times\delta^{z'}\times\delta^{y'}} \sim_{K^7} \delta^{-f_{\mathbf{X}}(x',y',z';x,y)}, \qquad (x',y',z') \in \mathbf{D}_{m'}. \tag{4.19}$$

By Lemma 4.5,

$$\frac{\mathcal{J}^+_{10\delta^t}(\mathbf{X}')}{10\delta^t|\mathbf{X}'|^2} \geq \frac{\mathcal{I}^+_{10\delta^t}(\mathbf{X}')}{10\delta^t|\mathbf{X}'|^2} \gtrsim_{K^5} \frac{|\mathbf{X}'|_{\delta^t\times\delta^t\times\delta^t}}{\delta^t|\mathbf{X}'|_{\delta^t\times\delta^t\times 1}|\mathbf{X}'|_{1\times\delta^t\times\delta^t}} \overset{(4.19)}{\sim_{K^{21}}} \delta^{-b_{\mathbf{X}}(t;x,y)}. \tag{4.20}$$

Here, and below, $\mathcal{J}^+$ is defined for some function $\chi \in C^\infty_c(\mathbb{R}^2)$ satisfying $\mathbf{1}_{B(1/2)} \leq \chi \leq \mathbf{1}_{B(1)}$; now the first inequality above follows from Lemma 3.4.

Recall the (small) absolute constant $a > 0$ defined above (4.17). Recall that $\delta = 2^{-mT}$, abbreviate $\bar\delta := \delta^{1-x-y}$, and write

$$\begin{aligned}\Big|\frac{\mathcal{J}^+_{10\delta^t}(\mathbf{X}')}{10\delta^t|\mathbf{X}'|^2} - \frac{\mathcal{J}^+_{a\bar\delta}(\mathbf{X}')}{a\bar\delta|\mathbf{X}'|^2}\Big| &\leq \Big|\frac{\mathcal{J}^+_{10\delta^t}(\mathbf{X}')}{10\delta^t|\mathbf{X}'|^2} - \frac{\mathcal{J}^+_{\delta^t}(\mathbf{X}')}{\delta^t|\mathbf{X}'|^2}\Big| + \Big|\frac{\mathcal{J}^+_{a\bar\delta}(\mathbf{X}')}{a\bar\delta|\mathbf{X}'|^2} - \frac{\mathcal{J}^+_{\bar\delta}(\mathbf{X}')}{\bar\delta|\mathbf{X}'|^2}\Big| \\ &\quad + \sum_{j=mt}^{m(1-x-y)-1} \Big|\frac{\mathcal{J}^+_{2^{-jT}}(\mathbf{X}')}{2^{-jT}|\mathbf{X}'|^2} - \frac{\mathcal{J}^+_{2^{-(j+1)T}}(\mathbf{X}')}{2^{-(j+1)T}|\mathbf{X}'|^2}\Big|.\end{aligned} \tag{4.21}$$

We will apply Corollary 4.15 to each of the terms with constants $A \in \{2^T, 1/10, 1/a\}$. Writing $2^{-jT} = \delta^{j/m}$, using (4.19), and estimating $j \leq m$,

$$\begin{aligned}\Big|\frac{\mathcal{J}^+_{2^{-jT}}(\mathbf{X}')}{2^{-jT}|\mathbf{X}'|^2} - \frac{\mathcal{J}^+_{2^{-(j+1)T}}(\mathbf{X}')}{2^{-(j+1)T}|\mathbf{X}'|^2}\Big| &\lesssim K2^{2T}\left(|\mathbf{X}'|^{-1}_{2^{-jT}\times 2^{-jT}\times 1}|\mathbf{X}'|^{-1}_{1\times 2^{-jT}\times 2^{-jT}}2^{3jT}\log 2^{jT}\right)^{1/2} \\ &\lesssim_{K^8} m2^{3T}\cdot\delta^{e_{\mathbf{X}}(j/m;x,y)}, \quad j \in \{mt,\ldots,m(1-x-y)-1\}.\end{aligned}$$

Similarly, using $\log \bar{\delta}^{-1} \leq \log \delta^{-1} = mT$,

$$\left| \frac{\mathcal{J}^{+}_{a\bar{\delta}}(\mathbf{X}')}{a\bar{\delta}|\mathbf{X}'|^2} - \frac{\mathcal{J}^{+}_{\bar{\delta}}(\mathbf{X}')}{\bar{\delta}|\mathbf{X}'|^2} \right| \lesssim Ka^{-2} \left(|\mathbf{X}|^{-1}_{\bar{\delta}\times\bar{\delta}\times 1} |\mathbf{X}|^{-1}_{1\times\bar{\delta}\times\bar{\delta}} \bar{\delta}^{-3} \log \bar{\delta}^{-1} \right)^{1/2}$$
$$\lesssim_{K^8} a^{-2} mT \cdot \delta^{e_{\mathbf{X}}(1-x-y;x,y)}.$$

Substituting these estimates back into (4.21), and also recalling (4.20), we find, for some absolute constants $c > 0$ and $C \geq 1$,

$$\tag{4.22} \begin{aligned} \frac{\mathcal{J}^{+}_{a\bar{\delta}}(\mathbf{X}')}{a\bar{\delta}|\mathbf{X}'|^2} &\geq cK^{-21}\delta^{-b_{\mathbf{X}}(t;x,y)} - CK^8 a^{-2} m 2^{3T} \sum_{j=mt}^{m(1-x-y)} \delta^{e_{\mathbf{X}}(j/m;x,y)} \\ &\geq \delta^{-b_{\mathbf{X}}(t;x,y)} \big(cK^{-21} - Cm^2 K^8 2^{3T} \delta^{b_{\mathbf{X}}(t;x,y)} \sup_{t\leq s\leq 1-x-y} \delta^{e_{\mathbf{X}}(s;x,y)} \big). \end{aligned}$$

This estimate determines the absolute constants c, C in (4.17). Since $(t; x, y)$ is a (c_1, c_2)-effective triple, above

$$\delta^{b_{\mathbf{X}}(t;x,y)} \sup_{t\leq s\leq 1-x-y} \delta^{e_{\mathbf{X}}(s;x,y)} \leq \delta^{c_1}.$$

Next, note that $2^{3T} = \delta^{-3/m} \leq \delta^{-3/m_0} \leq \delta^{-c_1/2}$ according to (4.16). Since furthermore $m \leq mT = \log \frac{1}{\delta}$, we find

$$Cm^2 K^8 2^{3T} \delta^{b_{\mathbf{X}}(t;x,y)} \sup_{t\leq s\leq 1-x-y} \delta^{e_{\mathbf{X}}(s;x,y)} \leq CK^8 \delta^{c_1/2} (\log \tfrac{1}{\delta})^2 \overset{(4.17)}{\leq} \tfrac{c}{2} K^{-21}.$$

Substituting this upper bound back into (4.22), and also using Lemma 3.4,

$$\tag{4.23} \frac{\mathcal{I}^{+}_{2a\delta^{1-x-y}}(\mathbf{X}')}{a\delta^{1-x-y}|\mathbf{X}'|^2} \gtrsim \frac{\mathcal{J}^{+}_{a\bar{\delta}}(\mathbf{X}')}{a\bar{\delta}|\mathbf{X}'|^2} \geq \tfrac{c}{2} K^{-21} \delta^{-b_{\mathbf{X}}(t;x,y)} \overset{(*)}{\geq} \tfrac{c}{2} K^{-21} \delta^{10c_2} \overset{(4.17)}{\geq} \delta^{11c_2}.$$

The estimate $(*)$, namely $\delta^{-b_{\mathbf{X}}(t;x,y)} \geq \delta^{10c_2}$, follows from the hypothesis (in the definition of effective triples) that $\max\{x, y, t\} \leq c_2$, and the Lipschitz property of the branching function $f_{\mathbf{X}}$, see [CPZ25, Lemma 3.8]. Here we also need T to be large enough in terms of c_2.

Next, as in the proof of [CPZ25, Proposition 4.2], we would like to show that "rescaling preserves incidences", and therefore (4.23) implies (4.15). There is a small technicality here (explained in Remark 4.17) which requires us to perform an additional argument. Unwrapping (4.23) (and estimating $\delta^{1-x-y} \geq \delta$), we have so far shown that

$$\tag{4.24} |\{(\omega_1, \omega_2) \in \mathbf{X}' \times \mathbf{X}' : \operatorname{dist}(z_{\omega_2}, \ell^{+}_{\omega_1}) \leq 2a\bar{\delta}\}| \gtrsim a\delta^{1+11c_2}|\mathbf{X}'|^2 \geq \delta^{1+24c_2}|\mathbf{X}|^2.$$

The final inequality follows from $|\mathbf{X}'| \gtrsim_{K^2} M_{\delta^x\times\delta^{x+y}\times\delta^y}(\mathbf{X}) \geq |\mathbf{X}|/|\mathbf{X}|_{\delta^x\times\delta^{x+y}\times\delta^y} \gtrsim \delta^{3(x+y)}|\mathbf{X}|$ and $\max\{x, y\} \leq c_2$, thus $a|\mathbf{X}'|^2 \geq \delta^{13c_2}|\mathbf{X}|^2$ if $T \geq 1$ is large enough in terms of c_2. For reasons to become apparent in a moment, we need to upgrade (4.24) to the following:

$$\tag{4.25} |\{(\omega_1, \omega_2) \in \mathbf{X}' \times \mathbf{X}' : x_{\omega_1} \leq x_{\omega_2} \text{ and } \operatorname{dist}(z_{\omega_2}, \ell^{+}_{\omega_1}) \leq 2a\bar{\delta}\}| \gtrsim \delta^{1+24c_2}|\mathbf{X}|^2,$$

where x_{ω_j} refers to the first coordinate of z_{ω_j}. To see that this can be done, note that if $x_{\omega_1} > x_{\omega_2}$ and $\mathrm{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \le 3a\bar{\delta}$, then $|z_{\omega_1} - z_{\omega_2}| \lesssim a\bar{\delta} = a\delta^{1-x-y}$. So, the difference set of the pairs in (4.24) and (4.25) is contained in the set of "nearby pairs"

$$\mathcal{N} := \{(\omega_1, \omega_2) \in \mathbf{X}' \times \mathbf{X}' : |z_{\omega_1} - z_{\omega_2}| \lesssim a\delta^{1-x-y}\}.$$

Further, for $\omega_1 \in \mathbf{X}'$ fixed and $a > 0$ small enough, the set $\{\omega_2 \in \mathbf{X}' : |z_{\omega_2} - z_{\omega_1}| \lesssim a\delta^{1-x-y}\}$ is contained in $\mathbf{X}' \cap \mathbf{R}_{\delta^{1-x-y}\times\delta^{1-x-y}\times 1}(\omega_1)$. Therefore, by the K-uniformity of $\mathbf{X}'$,

$$|\mathcal{N}| \le |\mathbf{X}'| M_{\delta^{1-x-y}\times\delta^{1-x-y}\times 1}(\mathbf{X}') \lesssim_K \frac{|\mathbf{X}|^2}{|\mathbf{X}'|_{\delta^{1-x-y}\times\delta^{1-x-y}\times 1}} \overset{(4.19)}{\lesssim_{K^7}} \frac{|\mathbf{X}|^2 |\mathbf{X}|_{\delta^x\times\delta^{x+y}\times\delta^y}}{|\mathbf{X}|_{\delta^{1-y}\times\delta\times\delta^y}}.$$

Here moreover $|\mathbf{X}|_{\delta^x\times\delta^{x+y}\times\delta^y} \le |\mathbf{X}|_{\delta^{2c_2}\times\delta^{2c_2}\times\delta^{2c_2}} \lesssim \delta^{-6c_2}$, and

$$|\mathbf{X}|_{\delta^{1-y}\times\delta\times\delta^y} \ge |\mathbf{X}|_{\delta^{1-y}\times\delta\times 1} \gtrsim \delta^y |\mathbf{X}|_{\delta\times\delta\times 1} \ge \delta^{c_2}|\mathbf{X}|_{\delta\times\delta\times 1} \overset{(4.18)}{\ge} \delta^{-1-31c_2}.$$

Altogether, $|\mathcal{N}| \lesssim_{K^7} \delta^{1+25c_2}|\mathbf{X}|^2$. Comparing this with the lower bound (4.24), we conclude that (4.25) holds, provided T is large enough in terms of c_2.

We may now complete the proof of (4.15). Recall that $\mathbf{X}' \subset \psi_{\mathbf{R}}(\mathbf{X} \cap \mathbf{R})$, where $\mathbf{R} = \mathbf{R}_{\delta^x\times\delta^{x+y}\times\delta^y}(a_0, b_0, \sigma_0)$. Fix a pair $(\omega_1', \omega_2') = (\psi_{\mathbf{R}}(\omega_1), \psi_{\mathbf{R}}(\omega_2)) \in \mathbf{X}' \times \mathbf{X}'$ such that

$$x_{\omega_1'} \le x_{\omega_2'} \quad \text{and} \quad \mathrm{dist}(z_{\omega_2'}, \ell^+_{\omega_1'}) \le 2a\delta^{1-x-y}.$$

It is shown at the end of the proof of [CPZ25, Proposition 4.2] that $\mathrm{dist}(z_{\omega_2}, \ell_{\omega_1}) \le \delta$, provided that $a > 0$ is a small enough (absolute) constant. To show *a fortiori* that $\mathrm{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \le \delta$, it suffices to demonstrate that $x_{\omega_1} \le x_{\omega_2}$. Write $\omega_j = (a_j, b_j, \sigma_j)$ for $j \in \{1, 2\}$. With this notation $x_{\omega_j} = a_j$, and (recalling the form of $\psi_{\mathbf{R}}$ from Definition 4.11),

$$\frac{a_1 - a_0}{\delta^x} = x_{\omega_1'} \le x_{\omega_2'} = \frac{a_2 - a_0}{\delta^x}. \tag{4.26}$$

Therefore $x_{\omega_1} \le x_{\omega_2}$, as desired. Now (4.25) implies that $\mathcal{I}^+_\delta(\mathbf{X}) \ge \delta^{1+24c_2}|\mathbf{X}|^2$, which is stronger than (4.15). The proof of Proposition 4.16 is complete. □

Remark 4.17. Why did we need to upgrade (4.24) to (4.25)? Assume that $(\omega_1', \omega_2') = (\psi_{\mathbf{R}}(\omega_1), \psi_{\mathbf{R}}(\omega_2))$ is a pair satisfying $\mathrm{dist}(z_{\omega_2'}, \ell^+_{\omega_1'}) \le \delta^{1-x-y}$ (but perhaps not $x_{\omega_1'} \le x_{\omega_2'}$). From the information $\mathrm{dist}(z_{\omega_2'}, \ell^+_{\omega_1'}) \le \delta^{1-x-y}$ alone we may infer that $x_{\omega_2'} \ge x_{\omega_1'} - \delta^{1-x-y}$. Using the relation (4.26) this yields $x_{\omega_2} \ge x_{\omega_1} - \delta^{1-y}$. Since $y > 0$, this information is weaker than $x_{\omega_2} \ge x_{\omega_1} - \delta$, which is what we would need to conclude $\mathrm{dist}(z_{\omega_2}, \ell^+_{\omega_1}) \le \delta$.

4.4. Completing the proof. With Proposition 4.16 in hand, the rest of the proof of Theorem C is the same as the proof of [CPZ25, Theorem 1.9] (or Theorem 1.10). We recap the main steps. First, following [CPZ25, Section 3.9], one defines the space $\mathcal{L}$ of *limiting branching functions* $f\colon \mathbf{D} \to [0, \infty)$, where $\mathbf{D} = \{(x, y, z) : x, y \in [0, 1] \text{ and } z \in [0, \min\{1, x + y\}]\}$. We will not need the specifics of the definition here, so we refer the reader to [CPZ25, Section 3.9].

Definition 4.18 (Effective triples for limiting branching functions). Let $c_1 \geq 0$, $c_2 \geq 0$, and $f \in \mathcal{L}$. A triple $(t; x, y)$ with $0 \leq t \leq 1 - (x + y)$ is (c_1, c_2)*-effective for* f if $\max\{t, x, y\} \leq c_2 \leq \frac{1}{3}$, and

$$b(t; x, y) + e(s; x, y) > c_1, \qquad t \leq s \leq 1 - (x + y).$$

The functions b, e are defined by the same formulae as in (4.10) and (4.11).

This definition of effective triples is taken from [CPZ25, Section 4.3]. Next, we follow the notation in [CPZ25, Section 4.3] verbatim to define the family $\mathcal{L}^{\mathrm{good}} \subset \mathcal{L}$. This subset consists of the functions $f \in \mathcal{L}$ such that for every $c_2 > 0$, there exist $c_1 > 0$ and $(t; x, y)$ such that $(t; x, y)$ is a (c_1, c_2)-effective triple for f. In other words, f has effective triples for arbitrarily small values of "c_2" while also retaining $c_1 > 0$.

Next, following [CPZ25, Section 5.1], we define $\mathcal{L}_{\alpha,\beta} \subset \mathcal{L}$, for $\alpha, \beta \geq 0$, to consist of the functions $f \in \mathcal{L}$ such that $f(x, y, x + y) \geq \alpha x + \beta y$ for all $(x, y) \in \mathbf{D}$. With this notation, [CPZ25, Theorem 5.1] (one of the main results in [CPZ25]) is the following statement:

Theorem 4.19. *If* $\alpha, \beta \in (1, 2]$ *with* $\alpha + \beta > 3$, *then* $\mathcal{L}_{\alpha,\beta} \subset \mathcal{L}^{\mathrm{good}}$.

Next, we state [CPZ25, Proposition 4.4] for ray incidences:

Proposition 4.20. *Let* $\{(\mathbf{X}_k, \delta_k)\}_{k=1}^{\infty}$ *be a sequence, where each* $\mathbf{X}_k \subset \Omega$ *is a finite set,* $\delta_k \searrow 0$, *and every limiting branching function for* $\{(\mathbf{X}_k, \delta_k)\}_{k=1}^{\infty}$ *is in* $\mathcal{L}^{\mathrm{good}}$. *(The precise meaning of this final hypothesis is clarified above [CPZ25, Proposition 4.4], but we do not need the details here.) Then, for every* $\varepsilon > 0$ *there exists* $k_0(\varepsilon) \in \mathbb{N}$ *such that*

$$\mathcal{I}^+_{\delta_k}(\mathbf{X}_k) \geq \delta_k^{1+\varepsilon}|\mathbf{X}_k|^2, \qquad k \geq k_0(\varepsilon).$$

The proof of Proposition 4.20 is the same as the proof of [CPZ25, Proposition 4.4], and the difference between $\mathcal{I}^+_{\delta_k}(\mathbf{X}_k)$ and $\mathcal{I}_{\delta_k}(\mathbf{X}_k)$ has no effect on the argument (except that one needs to apply Proposition 4.16 inside the proof in place of [CPZ25, Proposition 4.2]).

We now have all the ingredients in place to prove Theorem C. This short argument may be copied from [CPZ25] verbatim, and we do so for completeness.

Proof (of Theorem C). Fix $\alpha, \beta \in (1, 2]$ with $\alpha + \beta > 3$, and assume that the conclusion fails. Then, for some $\varepsilon > 0$ there exist sequences $\eta_k \to 0$ and $\delta_k \to 0$, and a sequence of Frostman $(\delta_k, \alpha, \beta, \delta_k^{-\eta_k})$-sets $\mathbf{X}_k \subset \Omega$ such that $\mathcal{I}^+_{\delta_k}(\mathbf{X}_k) \leq \delta_k^{1+\varepsilon}|\mathbf{X}_k|^2$ for all $k \in \mathbb{N}$. Any limiting branching function for $\{(\mathbf{X}_k, \delta_k)\}_{k=1}^{\infty}$ is in $\mathcal{L}_{\alpha,\beta} \subset \mathcal{L}^{\mathrm{good}}$ (the last inclusion being Theorem 4.19), so Proposition 4.20 produces a contradiction. □

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A. HIGH-LOW LEMMA FOR RAYS

In this section we prove Theorem 3.5, repeated below:

Theorem A.1. *Let $\mathbf{A}, \mathbf{B} \subset \Omega$ be finite sets. Then,*

$$\left|\frac{\mathcal{J}_w^+(\mathbf{A},\mathbf{B})}{w|\mathbf{A}||\mathbf{B}|} - \frac{\mathcal{J}_{w/2}^+(\mathbf{A},\mathbf{B})}{(w/2)|\mathbf{A}||\mathbf{B}|}\right| \leq A_\chi \Big(\frac{M_{1\times w}(\mathbf{A})M_{w\times w}(\mathbf{B})}{|\mathbf{A}||\mathbf{B}|} \cdot w^{-3}\log w^{-1}\Big)^{1/2}, \qquad w \in (0, \tfrac{1}{2}],$$

where the constant $A_\chi > 0$ depends only on χ.

Proof. The implicit constants in the $\lesssim$ notation in this argument may depend on χ. Abbreviate $B(w) := \mathcal{J}_w^+(\mathbf{A},\mathbf{B})/(|\mathbf{A}||\mathbf{B}|w)$. With this notation, using the definition (3.6),

$$B(w) - B(w/2) = \tfrac{1}{|\mathbf{A}||\mathbf{B}|}\langle \chi_{w/2} * (\chi_w * f_w - \chi_{w/4} * f_{w/2}), g\rangle.$$

Therefore, by Cauchy-Schwarz,

$$|B(w) - B(w/2)| \leq \tfrac{1}{|\mathbf{A}||\mathbf{B}|}\|\chi_w * f_w - \chi_{w/4} * f_{w/2}\|_{L^2([-2,2]^2)}\|\chi_{w/2} * g\|_{L^2([-2,2]^2)}. \tag{A.1}$$

(The integral may be restricted to $[-2,2]^2$ since $\operatorname{spt} g = P[\mathbf{B}] \subset [-1,1]^2$.) For the second factor, we use the definition of $M_{w\times w}(\mathbf{B})$ (see (3.7)) to deduce that

$$\|\chi_{w/2} * g\|_2^2 \lesssim \|g\|\|\chi_{w/2} * g\|_{L^\infty} \lesssim w^{-2} \cdot |\mathbf{B}|M_{w\times w}(\mathbf{B}). \tag{A.2}$$

To bound the first factor in (A.1), we define

$$\Phi_\omega^+ := \chi_w * \frac{\mathbf{1}_{[\ell_\omega^+]_{w/2}}}{w} - \chi_{w/4} * \frac{\mathbf{1}_{[\ell_\omega^+]_{w/4}}}{w/2}, \qquad \omega \in \mathbf{A}. \tag{A.3}$$

We also define a version of the Φ-function where ℓ_ω^+ is replaced by ℓ_ω:

$$\Phi_\omega := \chi_w * \frac{\mathbf{1}_{[\ell_\omega]_{w/2}}}{w} - \chi_{w/4} * \frac{\mathbf{1}_{[\ell_\omega]_{w/4}}}{w/2}, \qquad \omega \in \mathbf{A}. \tag{A.4}$$

With this notation, it holds

$$\chi_w * f_w - \chi_{w/4} * f_{w/2} = \sum_{\omega\in\mathbf{A}} \Phi_\omega^+.$$

Let $\psi \in C_c^\infty(\mathbb{R}^2)$ be a function with $\mathbf{1}_{[-2,2]^2} \leq \psi \leq \mathbf{1}_{[-3,3]^2}$. Then,

$$\|\chi_w * f_w - \chi_{w/4} * f_{w/2}\|_{L^2([-2,2]^2)}^2 \leq \sum_{\omega_1,\omega_2\in\mathbf{A}} \Big|\int \Phi_{\omega_1}^+\Phi_{\omega_2}^+\psi\Big|. \tag{A.5}$$

The most technical argument in [CPZ25, Appendix A.1] concerns the related integral $\int \Phi_{\omega_1}\Phi_{\omega_2}\psi$. It is proved (in an unnumbered display towards the end of [CPZ25, Appendix A.1]) that

$$\Big|\int \Phi_{\omega_1}\Phi_{\omega_2}\psi\Big| \lesssim \min\{w^{-1}, w^2/d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})^3\}. \tag{A.6}$$

This estimate is (as far as we can tell) not true with $\Phi_{\omega_j}^+$ in place of Φ_{ω_j}. However, it is true if the origin of ℓ_1 stays at distance $\sim w$ away from ℓ_2, and vice versa.

Claim I. *If* $\mathrm{dist}(z_{\omega_1}, \ell_{\omega_2}) \geq 20w$ *and* $\mathrm{dist}(z_{\omega_2}, \ell_{\omega_1}) \geq 20w$*, then*

$$\Big| \int \Phi^+_{\omega_1} \Phi^+_{\omega_2} \psi \Big| \lesssim \min\{w^{-1}, w^2/d_{\mathcal{A}}(\ell_{\omega_1}, \ell_{\omega_2})^3\}. \tag{A.7}$$

The proof is based on (A.6) and the following geometric observation:

Claim II. *Assume that* $\mathrm{dist}(z_{\omega_1}, \ell_{\omega_2}) \geq 20w$ *and* $\mathrm{dist}(z_{\omega_2}, \ell_{\omega_1}) \geq 20w$*. Then*

$$\Phi^+_{\omega_1} \Phi^+_{\omega_2} \equiv 0 \quad \textit{or} \quad \Phi^+_{\omega_1} \Phi^+_{\omega_2} \equiv \Phi_{\omega_1} \Phi_{\omega_2}. \tag{A.8}$$

Proof (of Claim I assuming Claim II). If the first alternative in (A.8) holds, then (A.7) is clear. If the second alternative holds, then

$$\Big| \int \Phi^+_{\omega_1} \Phi^+_{\omega_2} \psi \Big| = \Big| \int \Phi_{\omega_1} \Phi_{\omega_2} \psi \Big|,$$

and (A.7) is a consequence of (A.6). □

We then prove Claim II.

Proof (of Claim II). Abbreviate $\ell_{\omega_j} =: \ell_j$ and $\ell^+_{\omega_j} =: \ell^+_j$ and similarly $\Phi_{\omega_j} =: \Phi_j$ and $\Phi^+_{\omega_j} =: \Phi^+_j$. Let us also write

$$\ell_j = \{(x, a_j x + b_j) : x \in \mathbb{R}\} \quad \text{and} \quad \ell^+_j = \{a_j x + b_j : x \geq c_j\}$$

for some $a_j, c_j \in [-1, 1]$. With this notation $z_j := z_{\omega_j} = (c_j, a_j c_j + b_j)$.

Assume first that $\ell^+_1 \cap \ell^+_2 = \emptyset$. We claim that $\Phi^+_1 \Phi^+_2 \equiv 0$. Assume to the contrary that $(\Phi^+_1 \Phi^+_2)(x, y) > 0$ for some $(x, y) \in \mathbb{R}^2$. We may and will assume that $\max\{c_1, c_2\} = c_1$. Note that

$$\mathrm{spt}\, \Phi^+_j \subset \{(x', y') : x' \geq c_j - 3w \text{ and } |y' - (a_j x' + b_j)| \leq 3w\}, \qquad j \in \{1, 2\}.$$

Therefore $x \geq \max\{c_1, c_2\} - 3w = c_1 - 3w$, and

$$|h(x)| \leq 6w \quad \text{where} \quad h(x) := (a_1 x + b_1) - (a_2 x + b_2).$$

We first dispose of the case where $c_1 - 3w \leq x \leq c_1$. Since h is 2-Lipschitz, $|h(c_1)| \leq 12w$. This implies $\mathrm{dist}(z_1, \ell_2) < 20w$ contrary to hypothesis. So, we may assume that $x \geq c_1$.

Either $h(x) \geq 0$ or $h(x) < 0$, and the treatment of these cases is similar, so we only consider $h(x) \geq 0$. Note that $h' = a_1 - a_2$ is constant. Assume first that $h' \geq 0$, in which case $h(c_1) \leq h(x) \leq 6w$. There are now two options: either $h(c_1) \geq 0$ or $h(c_1) < 0$. In the first case $h(c_1) \in [0, h(x)] \subset [0, 6w]$, which implies that $\mathrm{dist}(z_1, \ell_2) < 20w$. In the opposite case $h(c_1) < 0$, there exists a point $\xi \in (c_1, x] = (\max\{c_1, c_2\}, x]$ with $h(\xi) = 0$. This gives $\ell^+_1 \cap \ell^+_2 \neq \emptyset$, again contrary to assumption.

Assume finally that $h' < 0$. Since $h(x) \geq 0$, there exists $\xi \geq x \geq \max\{c_1, c_2\}$ such that $h(\xi) = 0$. This means that $\ell^+_1 \cap \ell^+_2 \neq \emptyset$, again contrary to assumption. We have now reached a contradiction in all cases, and proved that $\ell^+_1 \cap \ell^+_2 = \emptyset$ implies $\Phi^+_1 \Phi^+_2 \equiv 0$.

Next, assume that $\ell_1^+ \cap \ell_2^+ \neq \emptyset$. In this case we will no longer use the hypothesis $\max\{c_1, c_2\} = c_1$. We claim that $\Phi_1^+\Phi_2^+ \equiv \Phi_1\Phi_2$. Assume to the contrary that there exists $(x, y) \in \mathbb{R}^2$ such that

$$(\Phi_1^+\Phi_2^+)(x, y) \neq (\Phi_1\Phi_2)(x, y).$$

Then either $\Phi_1^+(x, y) \neq \Phi_1(x, y)$ or $\Phi_2^+(x, y) \neq \Phi_2(x, y)$; let us assume that $\Phi_1^+(x, y) \neq \Phi_1(x, y)$. Then $x \leq c_1 + 3w$, since $\Phi_1^+(x', y') = \Phi_1(x', y')$ for $x' > c_1 + 3w$ and $y' \in \mathbb{R}$.

Recall that $h(x) = (a_1x + b_1) - (a_2x + b_2)$. Observe that $h(x) \leq 6w$, since otherwise $(\Phi_1^+\Phi_2^+)(x, y) = 0 = (\Phi_1\Phi_2)(x, y)$. We dispose of the special case where $c_1 \leq x \leq c_1 + 2w$. Then $h(c_1) \leq 12w$, and therefore $\operatorname{dist}(z_1, \ell_2) < 20w$ contrary to hypothesis. So, we may assume that $x < c_1$.

Since we assumed that $\ell_1^+ \cap \ell_2^+ \neq \emptyset$, and $x < c_1$, there exists $\xi \geq c_1 > x$ such that $h(\xi) = 0$. But now $h(c_1) \in [h(\xi), h(x)]$ (since h is monotone), and consequently $|h(c_1)| \leq 6w$. This implies that $\operatorname{dist}(z_1, \ell_2) < 20w$, contrary to hypothesis. We have now proved that $\ell_1^+ \cap \ell_2^+ \neq \emptyset$ implies $\Phi_1^+\Phi_2^+ \equiv \Phi_1\Phi_2$, and the proof of Claim II is complete.□

We proceed to estimate the right hand side of (A.5). Write

$$\mathbf{Bad} := \{(\omega_1, \omega_2) \in \mathbf{A} \times \mathbf{A} : \operatorname{dist}(z_{\omega_1}, \ell_{\omega_2}) \leq 20w \text{ or } \operatorname{dist}(z_{\omega_2}, \ell_{\omega_1}) \leq 20w\}.$$

Also set $\mathbf{Bad}(\omega_1) := \{\omega_2 \in \mathbf{A} : (\omega_1, \omega_2) \in \mathbf{Bad}\}$, $\omega_1 \in \mathbf{A}$. Fixing $\omega_1 \in \mathbf{A}$, and using Claim I,

$$\begin{aligned}\sum_{\omega_2 \in \mathbf{A} \setminus \mathbf{Bad}(\omega_1)} \left| \int \Phi_{\omega_1}^+ \Phi_{\omega_2}^+ \psi \right| &\lesssim w^{-1} |\{\omega_2 \in \mathbf{A} : \ell_{\omega_2} \in B_{d_{\mathcal{A}}}(\ell_{\omega_1}, w)\}| \\ &\quad + \sum_{w \leq 2^{-j} \leq 100} w^2 2^{3j} |\{\omega_2 \in \mathbf{A} : \ell_{\omega_2} \in B_{d_{\mathcal{A}}}(\ell_{\omega_1}, 2^{-j})\}| \\ &\leq w^{-1} M_{1\times w}(\mathbf{A}) + w^2 \sum_{w \leq 2^{-j} \leq 100} 2^{3j} M_{1 \times 2^{-j}}(\mathbf{A}).\end{aligned}$$

Note that since $B_{d_{\mathcal{A}}}(\ell_0, r)$ can be covered by $\lesssim (r/w)^2$ many w-balls for $r \geq w$, it holds $M_{1\times r}(\mathbf{A}) \lesssim (r/w)^2 M_{1\times w}(\mathbf{A})$. Inputting this back into the estimate above,

$$\sum_{\omega_2 \in \mathbf{A} \setminus \mathbf{Bad}(\omega_1)} \left| \int \Phi_{\omega_1}^+ \Phi_{\omega_2}^+ \psi \right| \lesssim w^{-1} M_{1\times w}(\mathbf{A}) + \sum_{w \leq 2^{-j} \leq 100} 2^j M_{1\times w}(\mathbf{A}) \lesssim w^{-1} M_{1\times w}(\mathbf{A}).$$

Summing over $\omega_1 \in \mathbf{A}$, we therefore find

$$\sum_{(\omega_1, \omega_2) \in (\mathbf{A}\times\mathbf{A}) \setminus \mathbf{Bad}} \left| \int \Phi_{\omega_1}^+ \Phi_{\omega_2}^+ \psi \right| \lesssim w^{-1} |\mathbf{A}| M_{1\times w}(\mathbf{A}). \tag{A.9}$$

We next aim for a similar bound for the sum over $(\omega_1, \omega_2) \in \mathbf{Bad}$, which we organise as follows:

$$\sum_{(\omega_1,\omega_2)\in\mathbf{Bad}}\Big|\int\Phi^+_{\omega_1}\Phi^+_{\omega_2}\psi\Big| \leq \sum_{\omega_1\in\mathbf{A}}\sum_{\substack{\omega_2\in\mathbf{A}\\ \operatorname{dist}(z_{\omega_1},\ell_{\omega_2})\leq 20w}}\Big|\int\Phi^+_{\omega_1}\Phi^+_{\omega_2}\psi\Big| + \sum_{\omega_2\in\mathbf{A}}\sum_{\substack{\omega_1\in\mathbf{A}\\ \operatorname{dist}(z_{\omega_2},\ell_{\omega_1})\leq 20w}}\Big|\int\Phi^+_{\omega_1}\Phi^+_{\omega_2}\psi\Big| =: \Sigma_1+\Sigma_2.$$

The bounds for Σ_1 and Σ_2 are symmetric, so we only consider Σ_1. Now we no longer have the "orthogonality bound" (A.7) available, so we will have to make do with the following "non-cancellative" bound:

$$\Big|\int\Phi^+_{\omega_1}\Phi^+_{\omega_2}\psi\Big| \lesssim \min\{w^{-1}, d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})^{-1}\}. \tag{A.10}$$

This follows from the observations that $\|\Phi^+_{\omega_1}\Phi^+_{\omega_2}\|_{L^\infty}\lesssim w^{-2}$, and if $d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})\geq w$, then

$$(\operatorname{spt}\Phi^+_{\omega_1})\cap(\operatorname{spt}\Phi^+_{\omega_2})$$

is contained in a rectangle of dimensions $\sim w\times(w/d_{\mathcal{A}}(\ell_1,\ell_2))$. For $d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})\leq w$, one relies on the trivial fact that $\operatorname{spt}(\Phi^+_{\omega_1}\psi)$ is contained in a rectangle of dimensions $\sim w\times 1$.

Using (A.10), one gets

$$\Sigma_1 \lesssim \sum_{\omega_1\in\mathbf{A}}\Big(\sum_{\substack{\omega_2\in\mathbf{A}\\ d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})\leq w}} w^{-1} + \sum_{\substack{\omega_2\in\mathbf{A}:d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})\geq w\\ \operatorname{dist}(z_{\omega_1},\ell_{\omega_2})\leq 20w}} d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})^{-1}\Big). \tag{A.11}$$

The first sum in brackets is evidently bounded by $w^{-1}M_{1\times w}(\mathbf{A})$. To bound the second sum, we make the following observation: the relevant family of lines

$$A(z_{\omega_1}) := \{\ell_2\in\mathcal{A}(2,1):\operatorname{dist}(z_{\omega_1},\ell_2)\leq 20w\},$$

viewed as a set of parameters in $[-1,1]^2$, is contained in the $O(w)$-neighbourhood of a certain line, determined by z_{ω_1}. We partition $A(z_{\omega_1})\setminus B_{d_{\mathcal{A}}}(\ell_{\omega_1},w)$ according to the distance between ℓ_{ω_1} and $\ell_2\in A(z_{\omega_1})$:

$$A(z_{\omega_1})\setminus B_{d_{\mathcal{A}}}(\ell_{\omega_1},w) =: \bigcup_{w\leq 2^{-j}\leq 1} A_j,$$

where $A_j := \{\ell_2\in A(z_{\omega_1}) : d_{\mathcal{A}}(\ell_{\omega_1},\ell_2)\in[2^{-j},2^{-j+1})\}$. In parameter space, A_j is contained in a rectangle of dimensions $\sim w\times 2^{-j}$. Therefore A_j may be covered by a family $\mathcal{B}_j$ of w-balls of cardinality $\sim(2^{-j}/w)$. Since $|\{\omega_2\in\mathbf{A}:\ell_{\omega_2}\in B\}|\leq M_{1\times w}(\mathbf{A})$ for each $B\in\mathcal{B}_j$, we obtain

$$\sum_{\substack{\omega_2\in\mathbf{A}:d_{\mathcal{A}}(\ell_{\omega_1},\ell_{\omega_2})\geq w\\ d(z_{\omega_1},\ell_{\omega_2})\leq 20w}} d(\ell_{\omega_1},\ell_{\omega_2})^{-1} \lesssim \sum_{w\leq 2^{-j}\leq 1} 2^j\cdot|\{\omega_2\in\mathbf{A}:\ell_{\omega_2}\in A_j\}|$$

$$\leq \sum_{w\leq 2^{-j}\leq 1} 2^j \sum_{B\in\mathcal{B}_j} |\{\omega_2\in\mathbf{A} : \ell_{\omega_2}\in B\}|$$
$$\lesssim (w^{-1}\log w^{-1})M_{1\times w}(\mathbf{A}).$$

Substituting this estimate back into (A.11) yields $\Sigma_1 \lesssim (w^{-1}\log w^{-1})|\mathbf{A}|M_{1\times w}(\mathbf{A})$, and therefore

$$\sum_{(\omega_1,\omega_2)\in\mathbf{Bad}} \left|\int \Phi^+_{\omega_1}\Phi^+_{\omega_2}\psi\right| \lesssim (w^{-1}\log w^{-1})|\mathbf{A}|M_{1\times w}(\mathbf{A}).$$

Combining this estimate with the bound (A.9) for the "good" line-pairs, we find

$$\sum_{\omega_1,\omega_2\in\mathbf{A}} \left|\int \Phi^+_{\omega_1}\Phi^+_{\omega_2}\psi\right| \lesssim (w^{-1}\log w^{-1})|\mathbf{A}|M_{1\times w}(\mathbf{A}).$$

Substituting this bound into (A.5),

$$\|\chi_w * f_w - \chi_{w/4} * f_{w/2}\|^2_{L^2([-2,2]^2)} \lesssim (w^{-1}\log w^{-1})|\mathbf{A}|M_{1\times w}(\mathbf{A}).$$

Finally, plugging this estimate and (A.2) back into (A.1) yields

$$|B(w) - B(w/2)| \lesssim \left(\frac{M_{1\times w}(\mathbf{A})M_{w\times w}(\mathbf{B})}{|\mathbf{A}||\mathbf{B}|}\cdot w^{-3}\log w^{-1}\right)^{1/2}.$$

This completes the proof of Theorem 3.5. □

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Tuomas Orponen
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland
Email: tuomas.t.orponen@jyu.fi

Alex Rutar
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 University of Jyväskylä, Finland
Email: alex@rutar.org