Geometric and Combinatorial Properties of Self-similar Measures

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Self-similar Measures and the Weak Separation Condition

Definition 0.1 *Let $\mathcal{I}$ be a finite index set and let $\{S_i\}_{i\in\mathcal{I}}$ be maps from $\mathbb{R}$ to $\mathbb{R}$ of the form*

$$S_i(x) = r_i x + d_i \text{ where } 0 < |r_i| < 1 \text{ and } d_i \in \mathbb{R}$$

for each $i \in \mathcal{I}$. Let $(p_i)_{i\in\mathcal{I}}$ be a probability vector, i.e. $p_i > 0$ and $\sum_{i\in\mathcal{I}} p_i = 1$. Then there is a unique Borel probability measure satisfying

$$\mu = \sum_{i\in\mathcal{I}} p_i \cdot \mu \circ S_i^{-1}.$$

We say μ is a self-similar measure, *and* $\operatorname{supp}\mu = K$ *is a* self-similar set.

Definition 0.2 *Let $\mathcal{I}^* = \bigcup_{k=0}^{\infty} \mathcal{I}^k$. Given $\sigma = (\sigma_1, \ldots, \sigma_j) \in \mathcal{I}^*$, we denote*

$$S_\sigma = S_{\sigma_1} \circ \cdots \circ S_{\sigma_j},\ r_\sigma = r_{\sigma_1} \cdots r_{\sigma_j},\ \text{and } p_\sigma = p_{\sigma_1} \cdots p_{\sigma_n}$$

We also write $\sigma^- = (\sigma_1, \ldots, \sigma_{j-1})$. Then given $t > 0$, put

$$\Lambda_t = \{\sigma \in \mathcal{I}^* : |r_\sigma| < t \le |r_{\sigma^-}|\}.$$

We say that $\{S_i\}_{i\in\mathcal{I}}$ satisfies the weak separation condition *if*

$$\sup_{x\in K, t>0} \#\{S_\sigma : \sigma \in \Lambda_r, S_\sigma(K) \cap B(x,t)\} < \infty.$$

Throughout, μ is a self-similar measure and the associated IFS satisfies the weak separation condition.

Net Intervals and Neighbour Sets

Let $h_1, \ldots, h_s$ be elements of the set $\{S_\sigma(0), S_\sigma(1) : \sigma \in \Lambda_t\}$ listed in strictly ascending order. An interval $[h_i, h_{i+1}]$ where $(h_i, h_{i+1}) \cap K \neq \emptyset$ is a *net interval* (of generation t).

Suppose Δ is a net interval. Denote by T_Δ the unique contraction $T_\Delta(x) = rx + a$ with $r > 0$ such that $T_\Delta(\operatorname{conv}(K)) = \Delta$.

Definition 0.3 *A similarity $f(x) = Rx + a$ is a* neighbour *of Δ of generation t if there exists some $\sigma \in \Lambda_t$ such that $S_\sigma(K) \cap \Delta^\circ \neq \emptyset$ and $f = T_\Delta^{-1} \circ S_\sigma$. The* neighbour set *of Δ is the set of all possible neighbours.*

We can define a notion of *transition generation*, $\operatorname{tg}(\Delta)$, to capture the notion of children of a neighbour set.

Proposition 0.4 *Up to rescaling by T_Δ, the geometry and neighbour sets of the children of a net interval Δ depend only on $\mathcal{V}(\Delta)$.*

The *transition graph* is a weighted graph where the vertices are the possible neighbour sets, and edges correspond to parent–child pairs of neighbour sets.

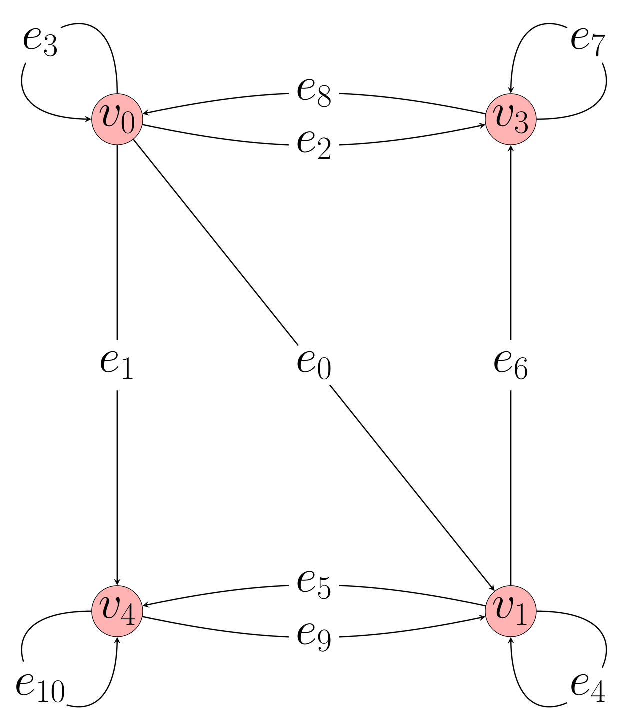
- Bijection π from finite paths in the transition graph with weight approximately t to net intervals in generation t.
- Can associate *transition matrices* to edges such that $\mu(\Delta)$ is the norm of the product of matrices corresponding to $\pi^{-1}(\Delta)$.
- The edge weights keep track of the current scale (important since the IFS is not necessarily equicontractive).

Example transition graph

Consider the IFS given by the maps

$$S_1(x) = \rho \cdot x \qquad S_2(x) = r \cdot x + \rho(1-r) \qquad S_3(x) = r \cdot x + 1 - r$$

where $\rho > 0$, $r > 0$ satisfy $\rho + 2r - \rho r \le 1$. The transition graph, along with the transition matrices, are given below:



Edge	Weight	Transition Matrix
e_0	ρ	(p_1)
e_1	r	$(p_1 p_3\ p_2)$
e_2	r	(p_2)
e_3	r	(p_3)
e_4	ρ	(p_1)
e_5	ρ	$(p_1 p_3\ p_2)$
e_6	r	(p_2)
e_7	r	(p_2)
e_8	r	(p_3)
e_9	r	$\begin{pmatrix} 1 \\ p_1 \end{pmatrix}$
e_{10}	r	$\begin{pmatrix} p_3 & 0 \\ p_1 p_3 & p_2 \end{pmatrix}$

Since the transition graph has only one loop class, the multifractal formalism is satisfied for all choices of probabilities.

L^q-spectra and Multifractal Formalism

Definition 0.5 *The* L^q-spectrum *of μ is given by*

$$\tau_\mu(q) := \liminf_{t\to 0} \frac{\log \sup \sum_i \mu(B(x_i,t))^q}{\log t}$$

for each $q \in \mathbb{R}$, where the supremum is over disjoint families of closed balls with centres $x_i \in K$.

Let

$$K_\mu(\alpha) = \Big\{x \in \operatorname{supp}\mu : \lim_{r\to 0} \frac{\log \mu(B(x,r))}{\log r} = \alpha\Big\}.$$

The *multifractal formalism* states, roughly speaking, that the dimension of the level sets can be computed as the concave conjugate of $\tau(q)$, i.e.

$$\dim_{\mathrm{H}} K(\alpha) = \tau_\mu^*(\alpha) := \inf_{q\in\mathbb{R}} \{q\alpha - \tau_\mu(q)\}.$$

If the IFS satisfies the open set condition, the multifractal formalism is always satisfied. However, under the weak separation condition, the multifractal formalism can fail for $q < 0$.

Definition 0.6 *We call the strongly connected components of the transition graph* loop classes. *We can associate to each loop class $\mathcal{L}$ a certain subadditive set function, and define corresponding* loop class L^q-spectra $\tau_{\mathcal{L}}$ *and* loop class multifractal spectra $f_{\mathcal{L}}$.

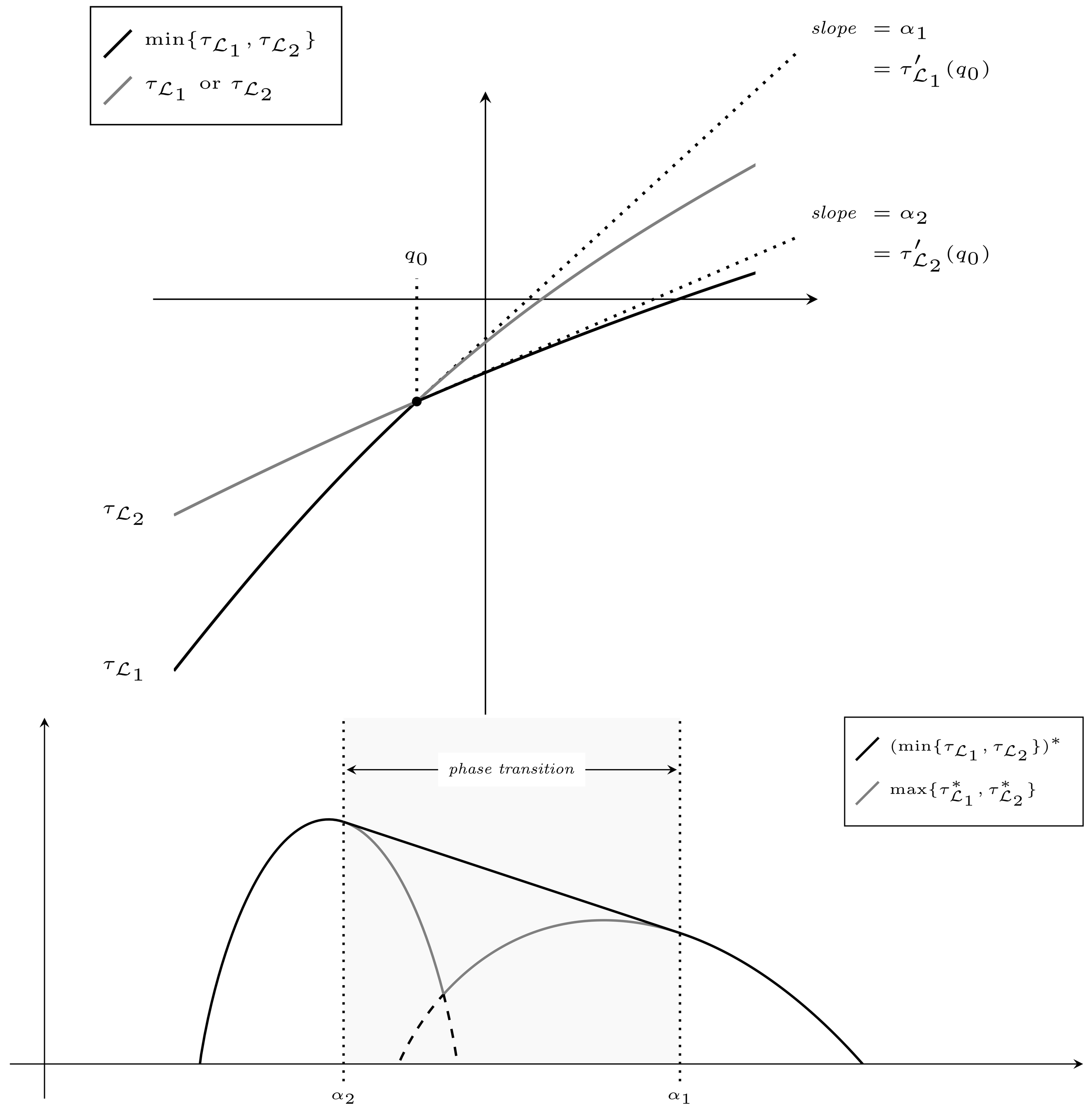
Heuristically, the loop classes will have "more regularity" than the self-similar measure μ, so one would hope that they satisfy the multifractal formalism.

Multifractal decomposition

Theorem 0.7 *Suppose μ is a self-similar measure with finite transition graph. Denote the loop classes by $\mathcal{L}_1, \ldots, \mathcal{L}_m$ and corresponding symbolic L^q-spectra $\tau_{\mathcal{L}_1}, \ldots, \tau_{\mathcal{L}_m}$. Suppose each loop class is non-degenerate. Then:*

1. *If the irreducibility assumption is satisfied,*
$$f_\mu(\alpha) = \max\{\tau_{\mathcal{L}_1}^*(\alpha), \ldots, \tau_{\mathcal{L}_m}^*(\alpha)\}.$$
2. *If the decomposability assumption is satisfied, the limit defining $\tau_\mu(q)$ exists for every $q \in \mathbb{R}$. Moreover,*
$$\tau_\mu(q) = \min\{\tau_{\mathcal{L}_1}(q), \ldots, \tau_{\mathcal{L}_m}(q)\}.$$

If the multifractal formalism fails, this occurs on the phase transitions between distinct loop class L^q-spectra.



References

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2. K. E. Hare, A. Rutar. *Local Dimensions of Self-similar Measures Satisfying the Finite Neighbour Condition.* arXiv:2101.07400 (submitted)
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