Attainable Forms of Intermediate Dimensions

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(joint w/ Amlan Banaji)

Dimensions?

- How does a set scale when you resize it?

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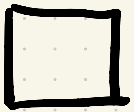
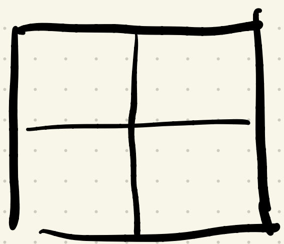
- 1



area
1

2



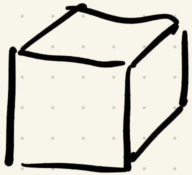
area
4

diameter $\times 2$

$\Downarrow$

area $\times 2^2$

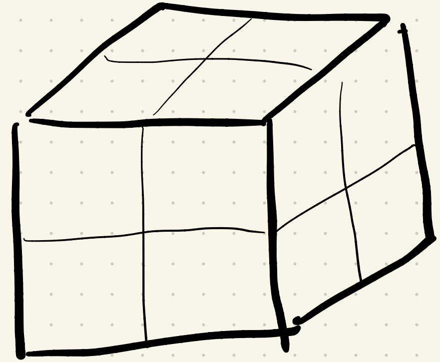
1



volume

1

8



volume

8

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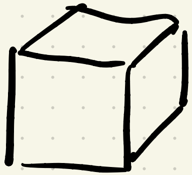
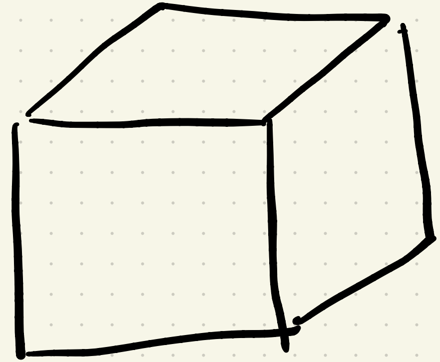
$\Downarrow$

volume $\times 2^3$

- 1 

  volume 1

  8 

  volume 8

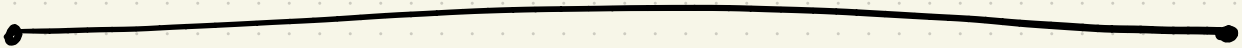
  diameter $\times 2$

  $\Downarrow$

  volume $\times 2^{3}$

- Square has dimension 2, cube has dimension 3.

Non-integer Dimension?



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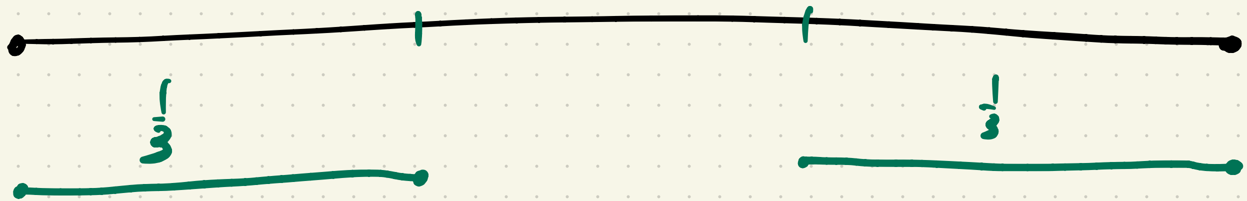

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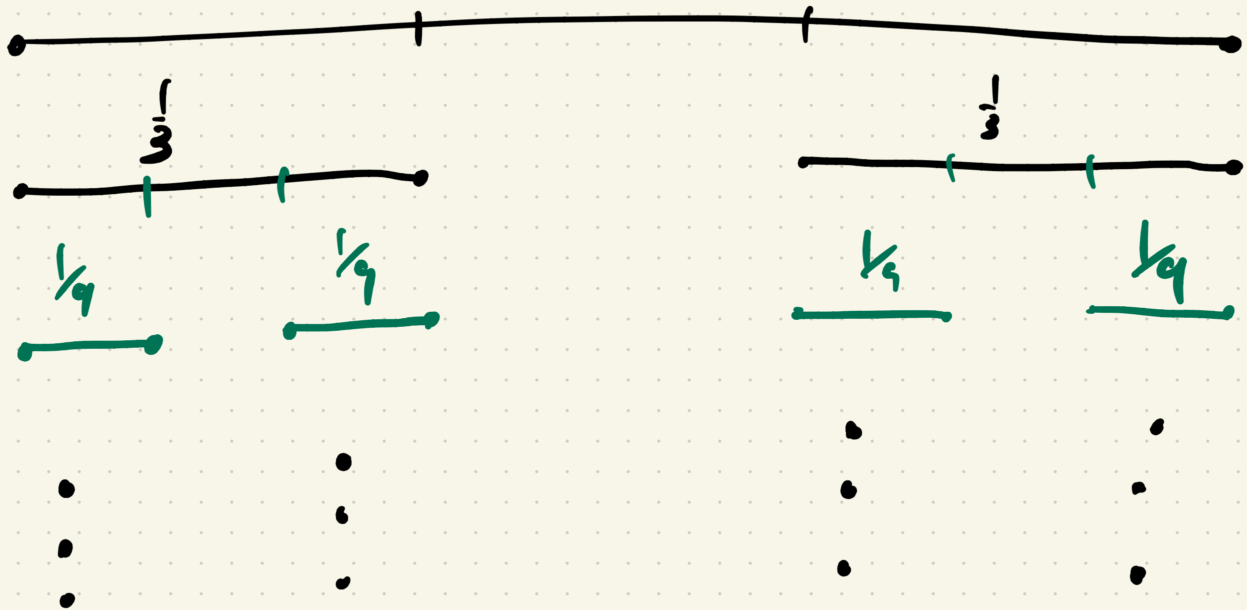

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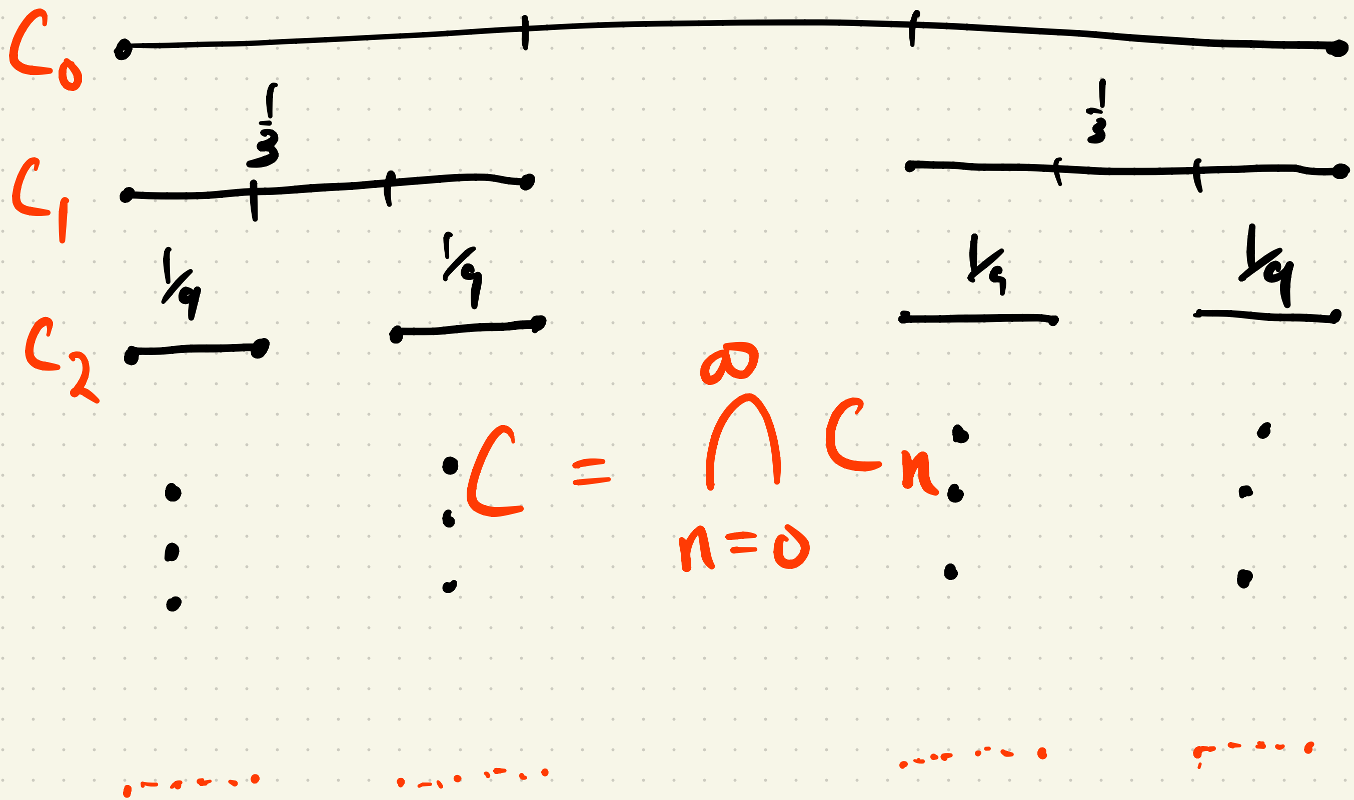
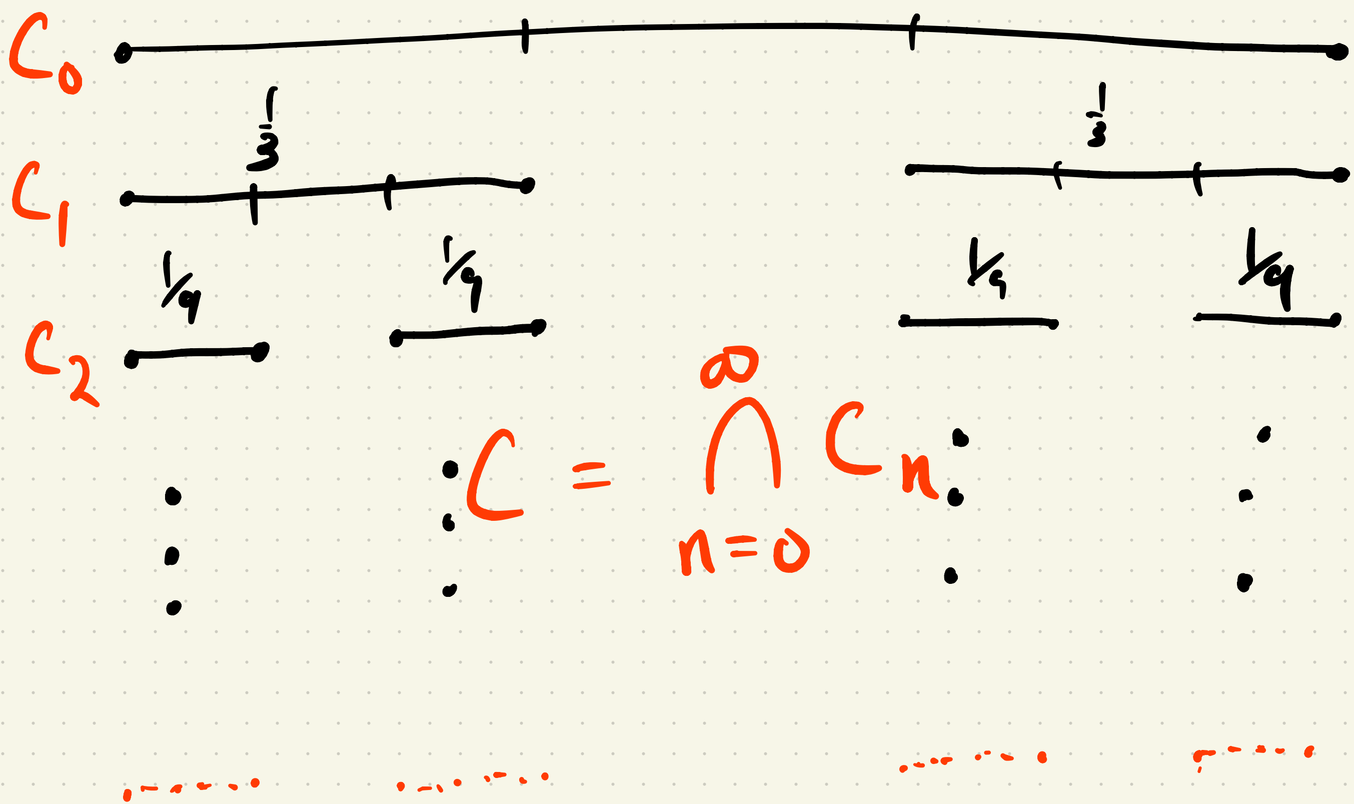


$$C = \bigcap_{n=0}^{\infty} C_n$$

Non-integer Dimension?



$$C = \bigcap_{n=0}^{\infty} C_n$$

Length?

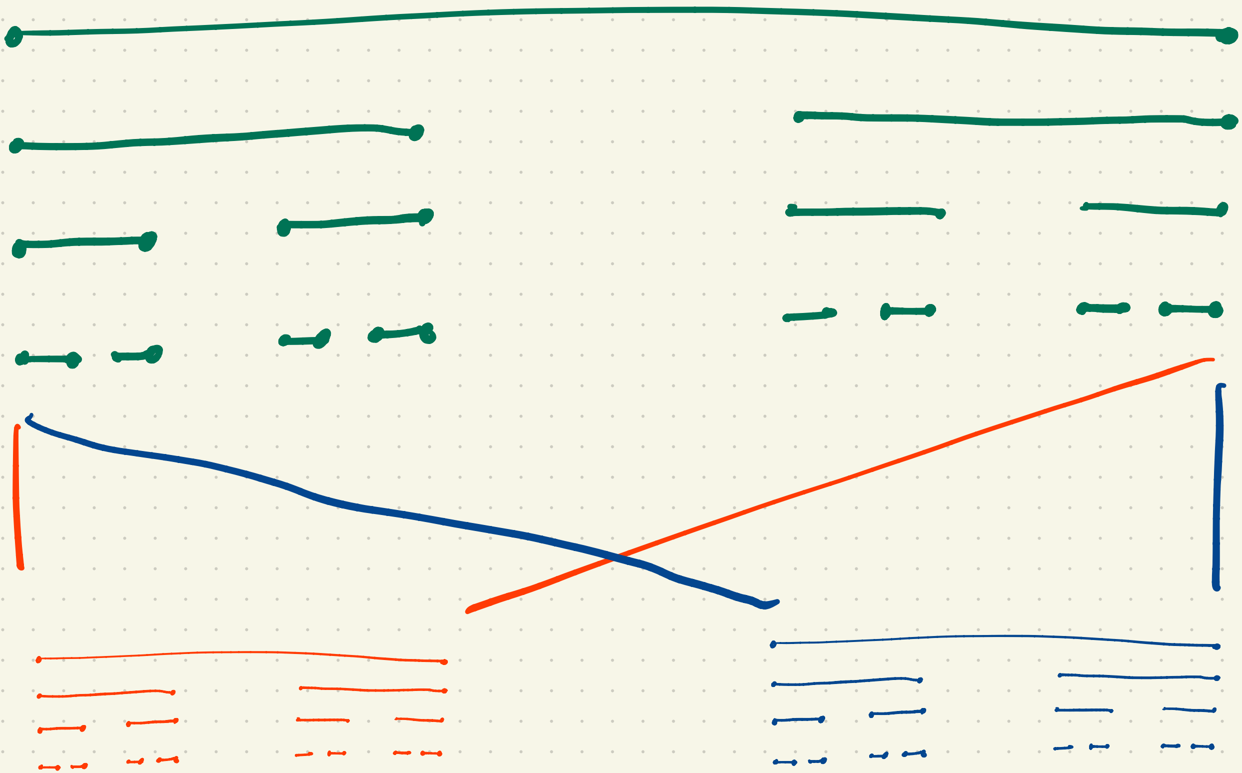
1

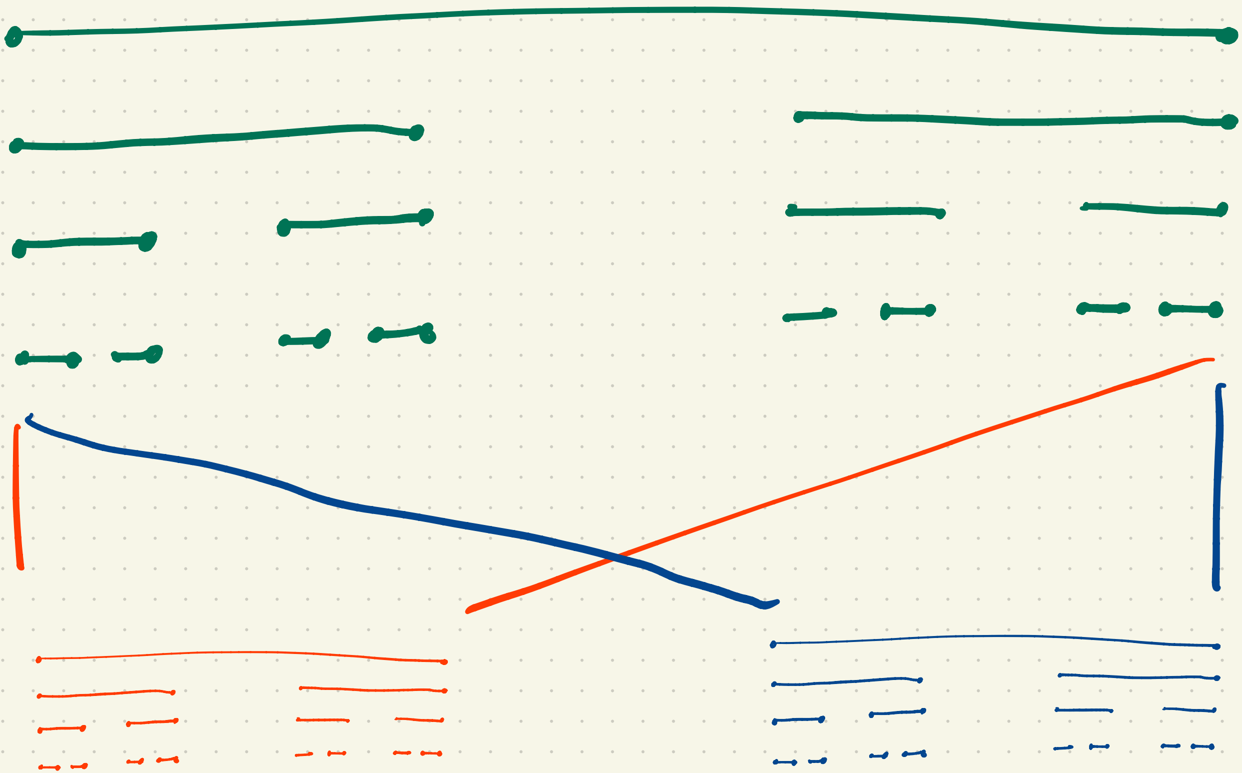
$\frac{2}{3}$

$\left(\frac{2}{3}\right)^2$

$\vdots$

0





$$C = \frac{1}{3}C + \frac{1}{3}\left(C + 2/3\right)$$

$$C = \frac{1}{3}C + \frac{1}{3}(C+2)$$

vs. interval L:

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scaling L by 2 $\Rightarrow$ change in length by 2

scaling C by 3 $\Rightarrow$ change in size? by 2

$$2^{\dim L} = 2 \implies \dim L = 1$$

$$3^{\dim C} = 2 \implies \dim C = \frac{\log 2}{\log 3}$$

non-integer dimension

Formalize "Size".

Recall: Lebesgue measure is unique Radon translation invariant measure on $\mathbb{R}$.

Lebesgue scales by power 1 under similarity (= isometry + scaling)

Can define Hausdorff s-measure:

$0 \leq s \leq 1$ as unique

- translation-invariant
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(not σ-finite)

Hausdorff dim = "correct scale s for $\mathcal{H}^s$-measure"

Box Dimension: $N_r(F) = \#$ intervals length r to cover F.

$$\overline{\dim}_B F = \limsup_{r \to 0} \frac{\log N_r(F)}{\log 1/r}$$

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$N_1 = 1$

$N_{\frac{1}{3}} = 2$

$N_{\frac{1}{9}} = 4$

$N_{\frac{1}{27}} = 8$

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$N_{\frac{1}{27}} = 8$

$$\overline{\dim}_B C = \limsup_{n\to\infty} \frac{\log N_{\frac{1}{3^n}}(C)}{\log 3^n}$$

$$= \lim_{n\to\infty} \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}.$$

Does $\overline{\dim}_B F = \dim_H F$ in general?

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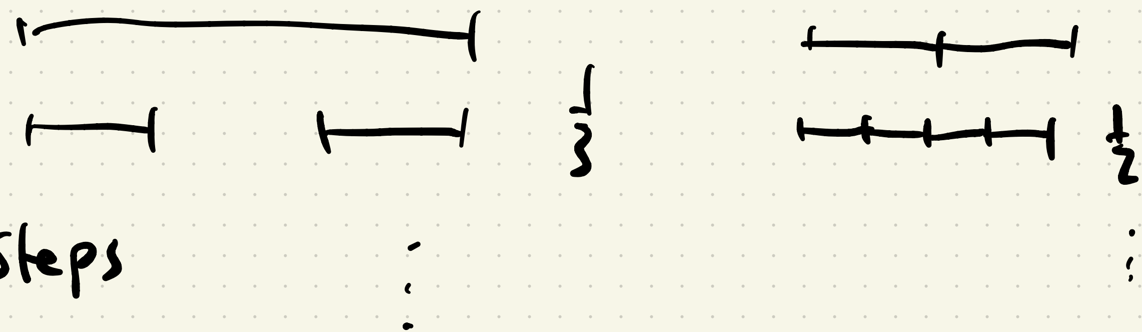
NO: consider Cantor construction



$\frac{1}{3}$ for N_1 steps

$\frac{1}{2}$ for N_2 steps

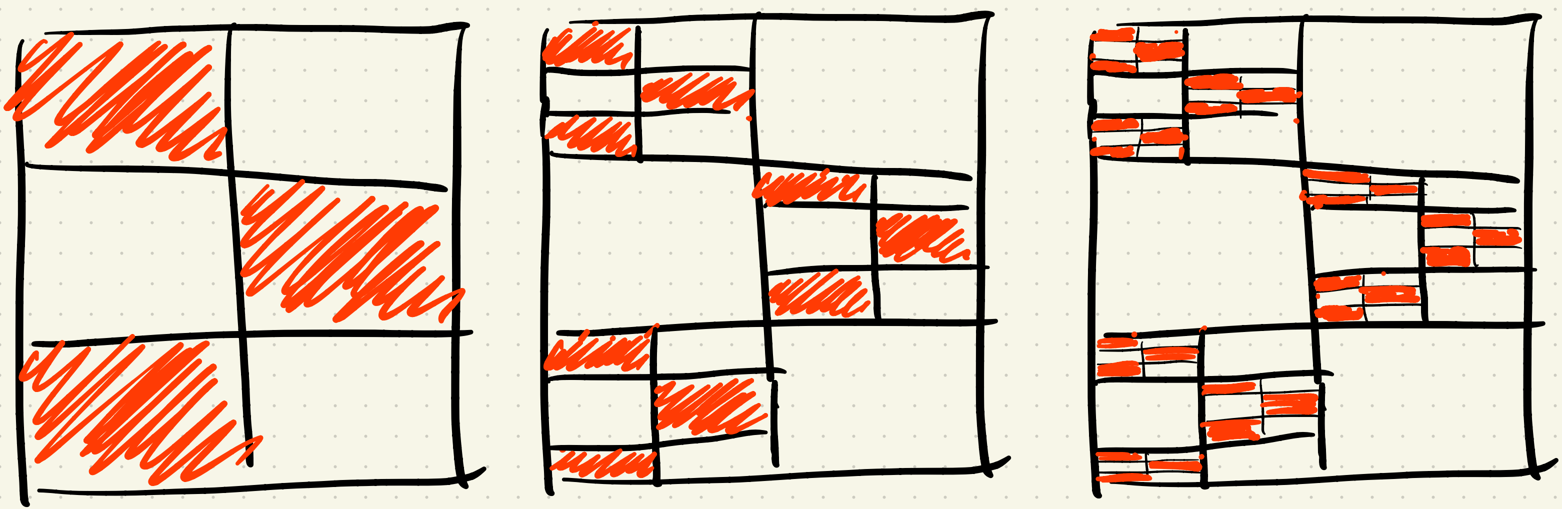
$\frac{1}{3}$ for N_3 steps etc.

$$N_1 \ll N_2 \ll N_3$$

$$\overline{\dim}_B M = 1 > \frac{\log 2}{\log 3} = \dim_H M.$$

this also happens for "natural"
(i.e. dynamically invariant) sets:

e.g. Bedford-McMullen Carpets



Intermediate Dimensions

$$\overline{\dim}_B K = \inf \{ s \geq 0 : \text{for all } \delta \text{ suff. small, } \exists \text{ cover } \{U_i\} \text{ of } K \text{ s.t.}$$

$$\cdot \ \operatorname{diam} U_i = \delta$$

$$\cdot \ \sum_i (\operatorname{diam} U_i)^s \leq 1 \}$$

Intermediate Dimensions

$$\overline{\dim}_\theta K = \inf\Big\{ s \geq 0 : \text{for all } \delta \text{ suff. small, } \exists \text{ cover } \{U_i\} \text{ of } K \text{ s.t.}$$

$$\cdot\ \delta^{\frac{1}{\theta}} \leq \operatorname{diam} U_i \leq \delta$$

$$\cdot\ \sum_i (\operatorname{diam} U_i)^s \leq 1 \Big\}$$

[for $\theta \in (0,1)$]

(note: also "lower" version)

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[for $\theta \in (0,1)$]

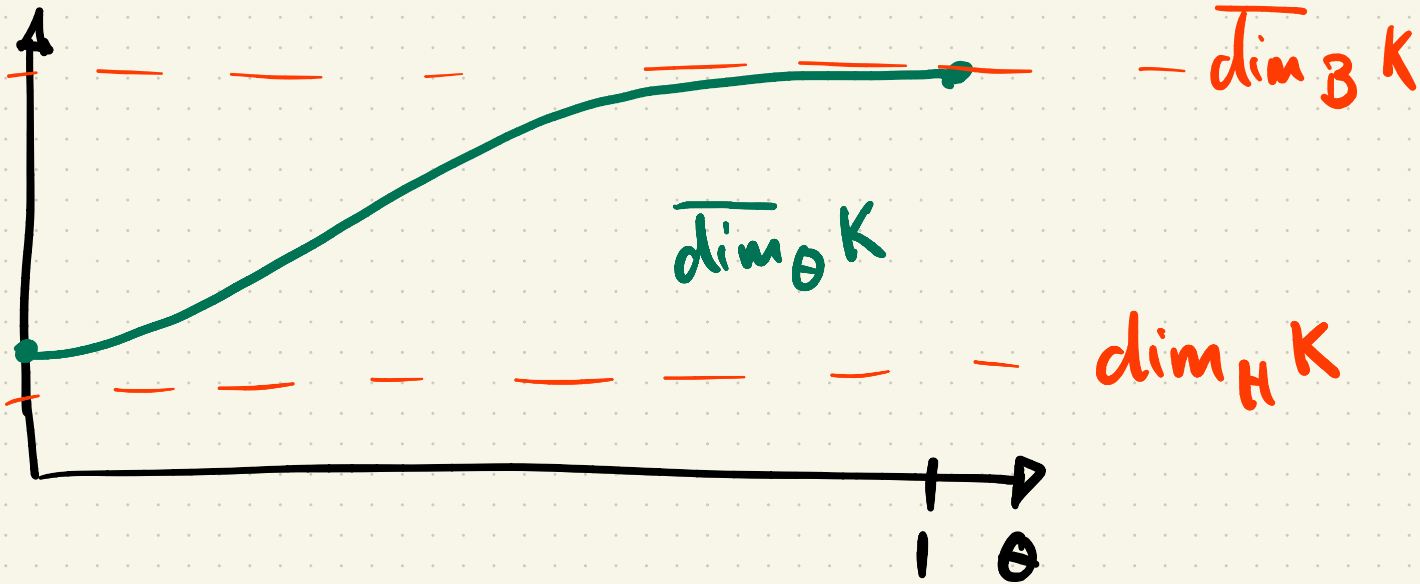
replace w/ 0:
get $\dim_H K$

(note: also "lower" version)

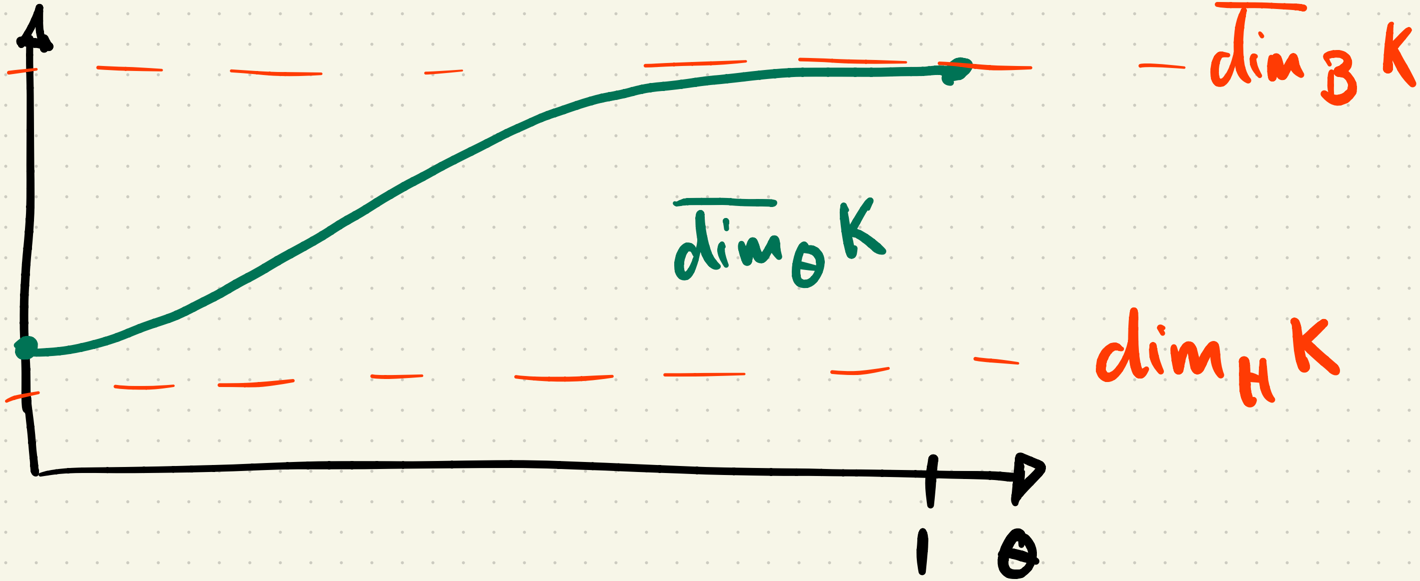
"Interpolate" b/w Box and Hausdorff dimension.

- $\overline{\dim}_\theta K$ is a continuous function of θ
- $\lim_{\theta \to 1} \overline{\dim}_\theta K = \overline{\dim}_B K$
- $\lim_{\theta \to 0} \dim_\theta K \geq \dim_H K$

proper is possible.



$\overline{\dim}_B K$
$\overline{\dim}_\theta K$
$\dim_H K$
1
θ



Question: how "rich" is the family of functions $\theta \mapsto \overline{\dim}_\theta K$?

Full Characterization for intermediate dimensions

Theorem (Banaji + AR): T.F.A.E

- $\exists K \subset \mathbb{R}^d$ s.t. $\overline{\dim}_\theta K = h(\theta)$
- $0 \leq D^+ h(\theta) \leq \dfrac{h(\theta)(d-h(\theta))}{d\theta}$

$$D^+ h(\theta) = \limsup_{\varepsilon \downarrow 0} \frac{h(\theta+\varepsilon) - h(\theta)}{\varepsilon}$$

Comments:

- if $f:(0,1)\to\mathbb{R}$ is increasing + Lipschitz, $\exists\, a>0, b\in\mathbb{R}, K\subset\mathbb{R}^d$

$$\dim_\theta K = a\cdot f(\theta) + b$$

(very general!)

Consequences:

- if $f:(0,1)\to\mathbb{R}$ is increasing + Lipschitz, $\exists\ a>0,\ b\in\mathbb{R},\ K\subset\mathbb{R}^d$
$$\dim_\theta K = a\cdot f(\theta) + b$$
(very general!)

- more general form simultaneously characterizing upper + lower, and accounting for Assouad + lower dimensions

Thank You!