#Alex

Dimension classification and branching functions

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Dimension Classification

Given family of dimensions $\{D_\tau\}_{\tau \in \Upsilon}$

what is the set of functions

$$\tau \mapsto D_\tau(E)$$

for some family of sets E ?

"Classical" exercise

- $\{\dim_H, \dim_P, \underline{\dim}_B, \overline{\dim}_B\}$
- $\{E : \emptyset \neq E \subseteq \mathbb{R}^d\}$

↳
$$
\begin{tikzcd}[column sep=small, row sep=small]
 & & \dim_P \arrow[dr, phantom, "\leqslant"] & & & \\
0 \leqslant & \dim_H \arrow[ur, phantom, "\leqslant"] \arrow[dr, phantom, "\leqslant"] & & \overline{\dim}_B & \leqslant & d \\
 & & \underline{\dim}_B \arrow[ur, phantom, "\leqslant"] & & &
\end{tikzcd}
$$

Intermediate dimensions — continuously parametrized dimensions for $0 < \theta \leq 1$

Theorem (Banaji–R., 2022) Classification for intermediate dimensions, with monotonicity and derivative constraint.

This talk: Assouad/lower spectrum

For $\theta \in (0,1)$

Assouad spectrum: infimum over $s \geq 0$ s.t.

$$\sup_{x \in E} N_r(B(x, r^\theta) \cap E) \lesssim_s \left(\frac{r^\theta}{r}\right)^s \quad (\forall\, 0<r<1)$$

Lower spectrum: supremum over $s \geq 0$ s.t.

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Notation $\dim_A^\theta E$; $\dim_L^\theta E$

Note: write $\overline{\varphi}_E(\theta) = (1-\theta)\cdot \dim_A^\theta E$

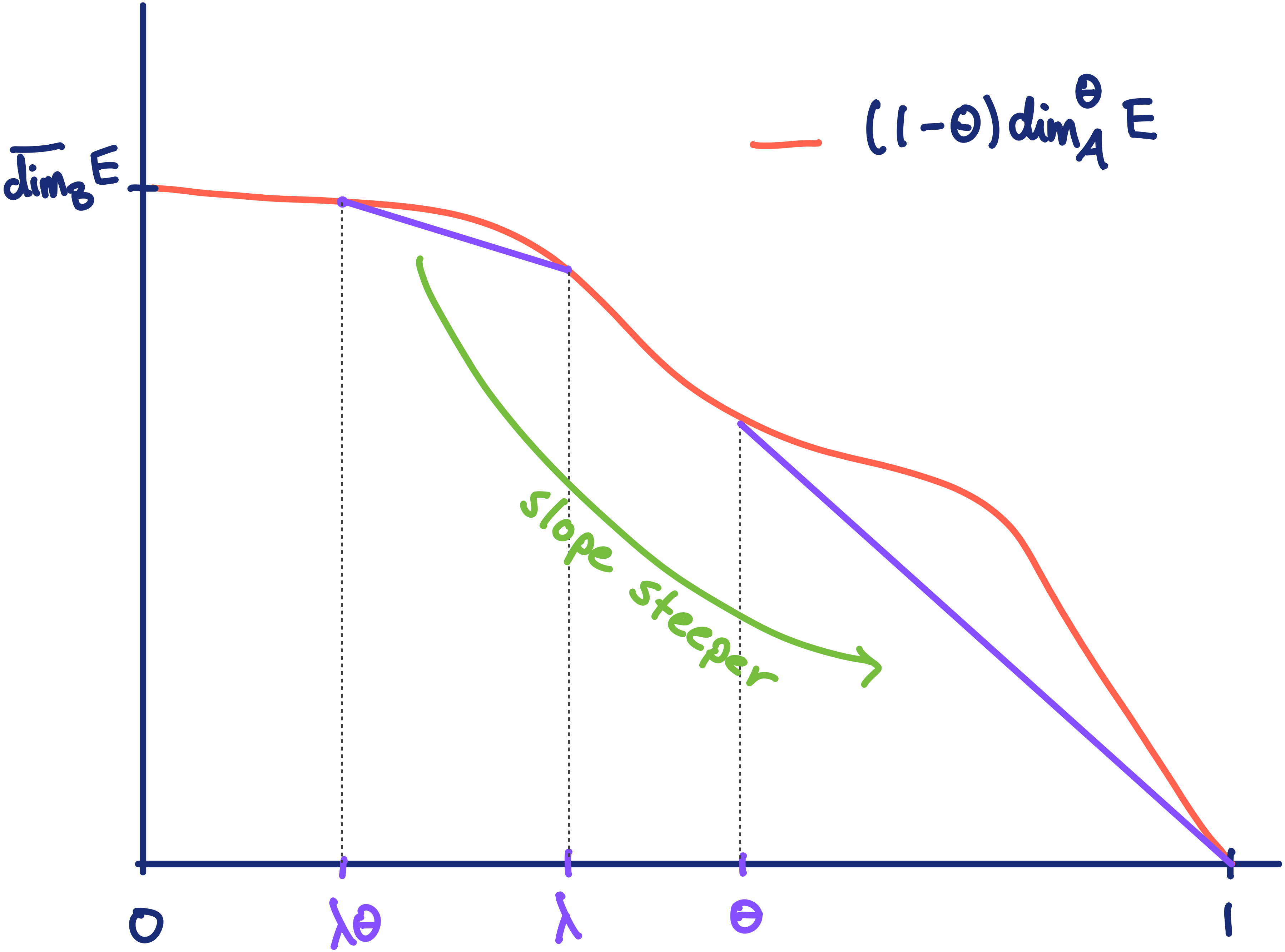
$\underline{\varphi}_E(\theta) = (1-\theta)\cdot \dim_L^\theta E$

Theorem (R. 2024)

$$\left\{ \theta \mapsto \overline{\varphi}_E(\theta) : E \subseteq \mathbb{R}^d \right\}$$

$$=$$

$$\left\{ \varphi : (0,1) \to [0,d] : \begin{array}{l} \text{a. } \varphi \text{ decreasing} \\ \text{b. } \varphi \text{ is } d\text{-Lipschitz} \\ \text{c. } \varphi(\lambda\theta) \leq \varphi(\theta) + \theta\varphi(\lambda) \\ \quad \text{for } \lambda, \theta \in (0,1) \end{array} \right\}$$


$(1-\theta)\dim_A^\theta E$
$\overline{\dim}_B E$
slope steeper
0
$\lambda\theta$
λ
θ
1

Construction technique for proof.

Given φ satisfying (a, b, c), Construct $E \subseteq \mathbb{R}^d$

s.t. $\varphi(\theta) = (1-\theta)\dim_A^{\theta} E$

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Let $f: \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ $\begin{cases} \text{increasing} \\ d\text{-Lipschitz} \\ f(0) = 0 \end{cases}$

- Define $a_k = f(k) - f(k-1) \in \{0, \dots, d\}$

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Base. $\mathcal{Q}_0 = \{[0,1]^d\}$

Ind. $Q \in \mathcal{Q}_{k-1} \longrightarrow 2^{a_k}$ cubes in $\mathcal{Q}_k$

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Define $a_k = f(k) - f(k-1) \in \{0, \dots, d\}$

$$K_f = \bigcap_{k=0}^{\infty} \bigcup_{Q \in \mathcal{Q}_k} Q$$

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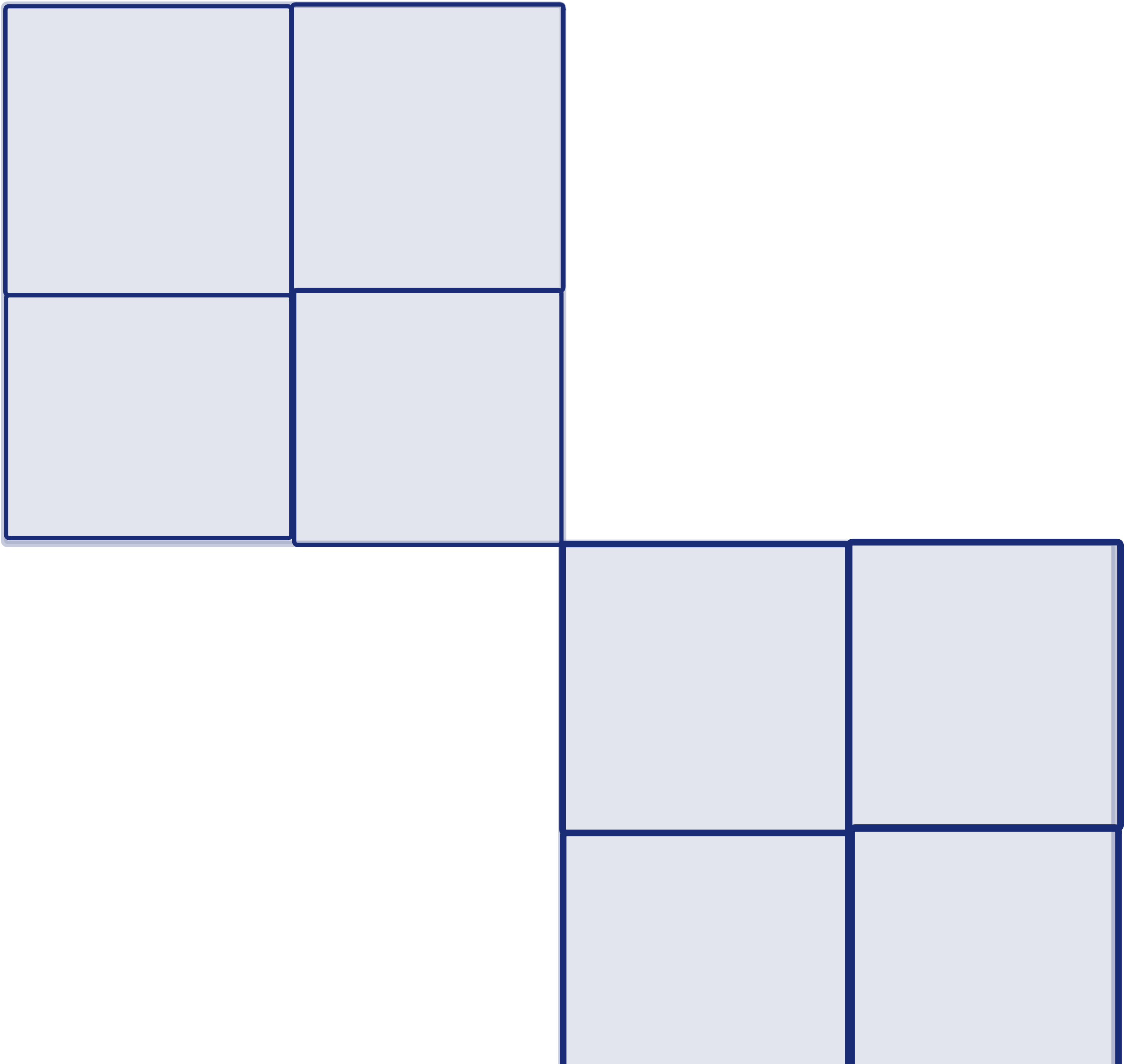
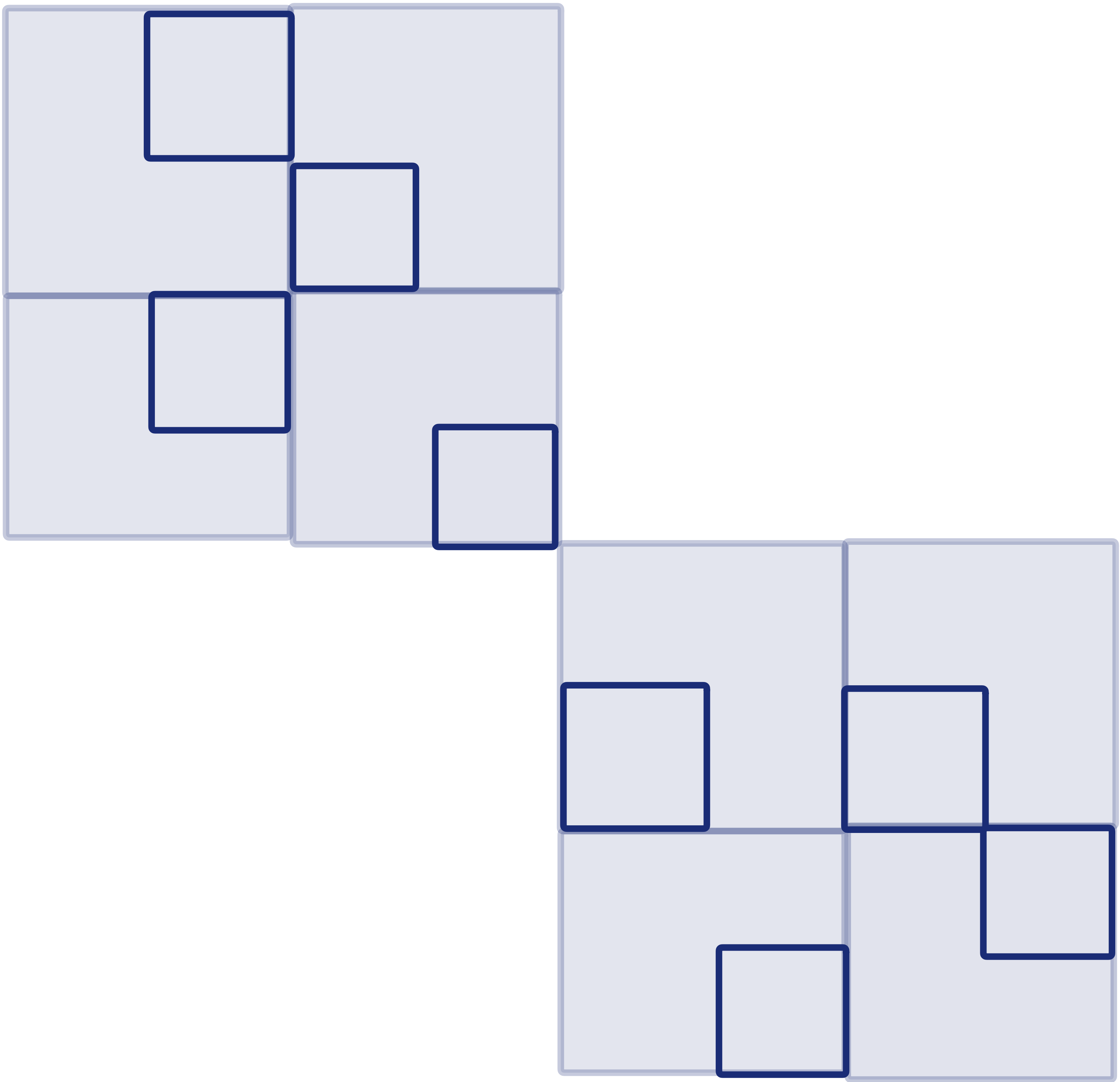
$a_1 = 1$



$a_1 = 1$

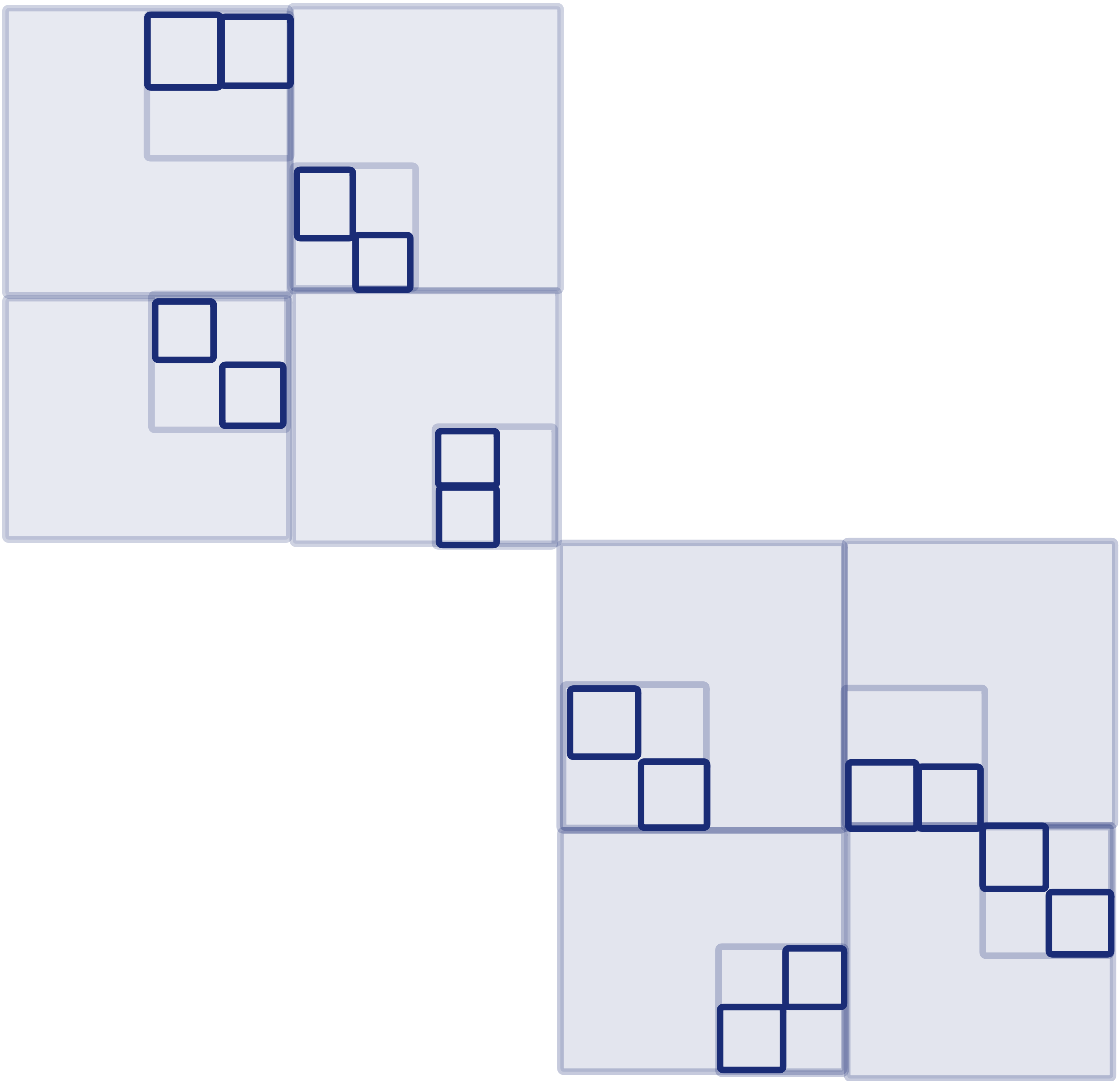
$a_2 = 2$



$a_1 = 1$

$a_2 = 2$

$a_3 = 0$



$a_1 = 1$

$a_2 = 2$

$a_3 = 0$

$a_4 = 1$

$\vdots$

etc

K_f "uniformly branching" set

$$(1-\theta)\dim_A^\theta K_f = \limsup_{u\to\infty} \frac{f(u)-f(\theta u)}{u}.$$

Choose f s.t. $\dfrac{f(u_k)-f(\theta u_k)}{u_k} = \varphi(\theta)$

for i.m. u_k, otherwise f constant

"no branching"

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"no branching"

f called the branching function of K_f.
(study of branching functions goes back to Bourgain; utility for general sets by Keleti–Shmerkin; ...?)

- All possible behaviour of $\dim_A^\theta E$ realized by uniformly branching set.

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- Also the case for intermediate dims

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What about lower spectrum?

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- $\inf_{x \in E} N_r(B(x,\frac{1}{2}) \cap E) \approx 1$

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$\rightsquigarrow$ proof technique for Assouad spectrum / intermediate dims does not work.

Two-scale branching functions

Two-scale branching functions.

$\beta : \Delta = \{(u,v) \in \mathbb{R}^2 : v \leq u\} \longrightarrow [0, \infty)$

$$\beta(u,v) = \log_2 \sup_{x \in E} N_{2^{-u}}(B(x, 2^{-v}) \cap E)$$

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$$\beta(u,v) = \log_2 \sup_{x \in E} N_{2^{-u}}\left(B(x,2^{-v}) \cap E\right)$$

Properties:

$$\left.\begin{array}{l} \cdot\ \beta(u,v) \text{ incr. in } u, \text{ decr in } v. \\ \cdot\ \beta(u,v) \text{ is } d\text{-Lipschitz in } u,v \\ \cdot\ \beta(u,v) \leq \beta(u,w) + \beta(w,v) \end{array}\right\} B(d)$$

to cover $B(x,2^{-v})$ at scale 2^{-u}, first cover at scale 2^{-w} then cover each ball at scale 2^{-u}

Theorem (Orgoványi–R., 2025+)

① If $E \subseteq \mathbb{R}^d$, $\exists f \in B(d)$ s.t. $f = \beta_E + O_d(1)$

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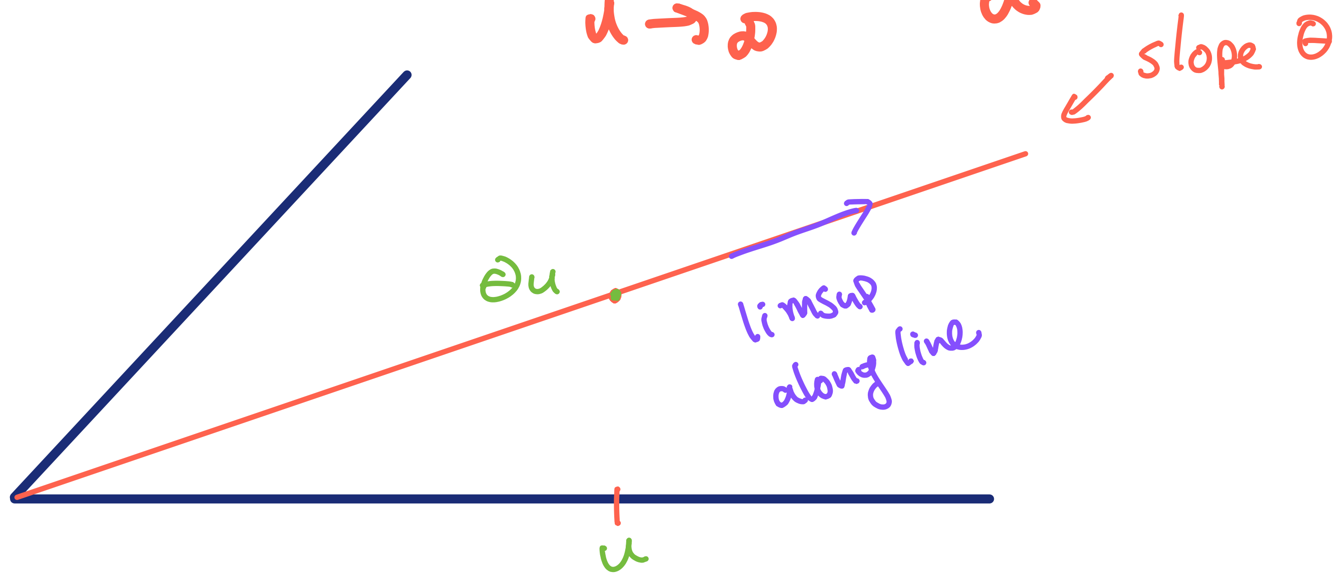
Implies classification of Assouad spectra since

$$(1-\theta)\dim_A^\theta E = \limsup_{u \to \infty} \frac{\beta_E(u, \theta u)}{u}$$



$$\varphi(\theta) = \limsup_{u \to \infty} \frac{\beta(u, \theta u)}{u}$$

$\varphi(\theta)$	$\beta(u,v)$
φ decreasing	$\beta(u,v)$ decr. in v
φ is d-Lipschitz	β is d-Lipschitz
$\varphi(\lambda\theta) \leq \varphi(\theta) + \theta\varphi(\lambda)$	$\beta(u,v) \leq \beta(u,w) + \beta(w,v)$
φ	$\beta(u,v) = u \cdot \varphi(v/u)$

Back to lower spectrum.

$$\gamma_E(u,v) = \log_2 \inf_{x\in E} P_{2^{-u}}(B(x,2^{-v})\cap E)$$

$P_{2^{-u}}$ = maximal $4\cdot 2^{-u}$-separated packing

Back to lower spectrum.

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$P_{2^{-u}}$ = maximal $4 \cdot 2^{-u}$- separated packing

Properties

- $\gamma_E(u,v) \geq \gamma_E(u,w) + \gamma_E(w,v)$

  $\Rightarrow$ incr u, decr v
- $\gamma_E(u,v)$ is d-Lipschitz in $\underline{\underline{u}}$

$\mathcal{L}(d)$

Recall $E = [0, \frac{1}{2}] \cup \{1\}$

$$\gamma_E(u, 0) = u \qquad + O(1)$$

$$\gamma_E(u, 1) = 0 \qquad + O(1)$$

NOT Lipschitz

Recall $E = [0, \tfrac{1}{2}] \cup \{1\}$

$$\gamma_E(u,0) = u \qquad + O(1)$$

$$\gamma_E(u,1) = 0 \qquad + O(1)$$

Not Lipschitz

γ_E [$\alpha(d)$]	β_E [$B(d)$]
[γ incr u, decr v]	β incr u, decr v
$\gamma(u,v) \geq \gamma(u,w) + \gamma(w,v)$	$\beta(u,v) \leq \beta(u,w) + \beta(w,v)$
γ d-Lipschitz in u	β d-Lipschitz in u, v

Theorem (Banaji – Chen – R. – Wang)

① If $E \subseteq \mathbb{R}^d$, $\exists f \in \mathcal{L}(d)$ s.t. $f \sim_d \gamma_E$

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Technical detail: $f \sim g$ if $\exists z \geq 0$

s.t. $f(u, v+z) - z \leq g(u,v)$

$f(u, v-z) + z \geq g(u,v)$

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Technical detail: $f \sim g$ if $\exists z \geq 0$ s.t. $f(u, v+z) - z \leq g(u,v)$
$f(u, v-z) + z \geq g(u,v)$

Similarly to Assouad spectrum:

$$(1-\theta)\dim_L^\theta E = \liminf_{u\to\infty} \frac{\gamma_E(u) - \gamma_E(\theta u)}{u}$$

↳ also resolve classification for lower spectrum

$\overline{\varphi}(\theta) = (1-\theta)\dim_A^\theta E$

$\overline{\varphi}$ decreasing

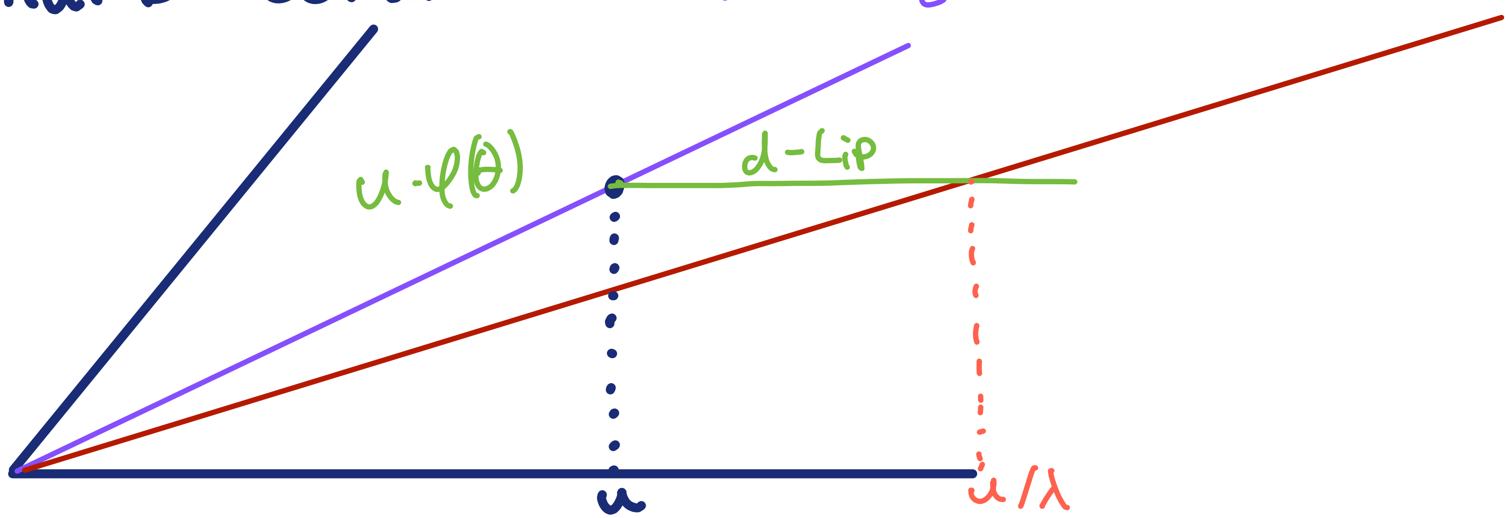
$\overline{\varphi}(\lambda\theta) \leqslant \overline{\varphi}(\theta) + \theta\overline{\varphi}(\lambda)$

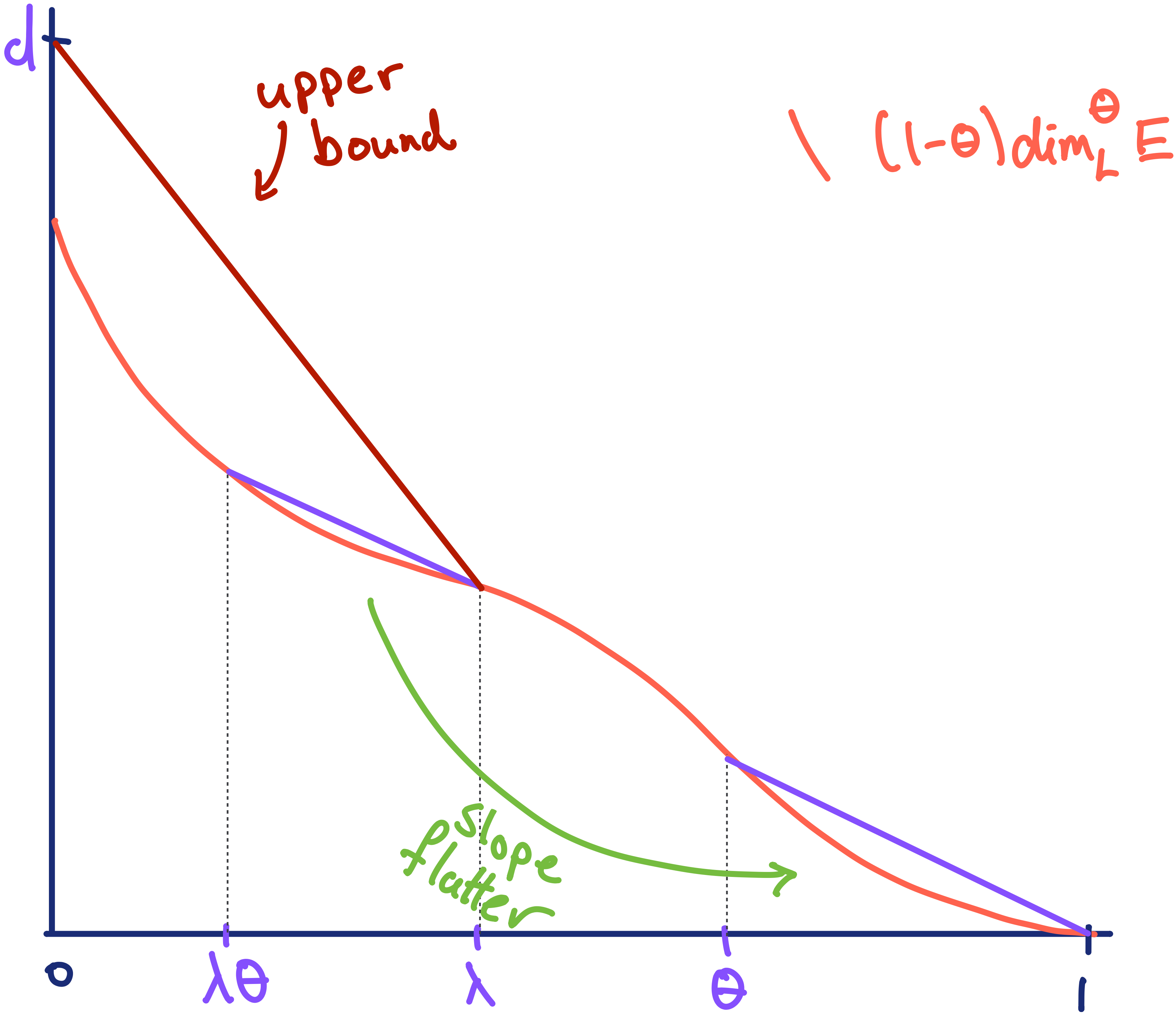
$\overline{\varphi}$ is d-Lipschitz

$\underline{\varphi}(\theta) = (1-\theta)\dim_L^\theta E$	$\overline{\varphi}(\theta) = (1-\theta)\dim_A^\theta E$
[$\underline{\varphi}$ decreasing] $\Uparrow$	$\overline{\varphi}$ decreasing
$\underline{\varphi}(\lambda\theta) \geq \underline{\varphi}(\theta) + \theta\underline{\varphi}(\lambda)$	$\overline{\varphi}(\lambda\theta) \leq \overline{\varphi}(\theta) + \theta\overline{\varphi}(\lambda)$
$\underline{\varphi}(\lambda\theta) \leq \theta \cdot \underline{\varphi}(\lambda) + (1-\theta) \cdot d$	$\overline{\varphi}$ is d-Lipschitz

$\underline{\varphi}(\theta) = (1-\theta)\dim_L^\theta E$	$\overline{\varphi}(\theta) = (1-\theta)\dim_A^\theta E$
[$\underline{\varphi}$ decreasing] $\Uparrow$	$\overline{\varphi}$ decreasing
$\underline{\varphi}(\lambda\theta) \geqslant \underline{\varphi}(\theta) + \theta\underline{\varphi}(\lambda)$	$\overline{\varphi}(\lambda\theta) \leqslant \overline{\varphi}(\theta) + \theta\overline{\varphi}(\lambda)$
$(*)\ \underline{\varphi}(\lambda\theta) \leqslant \theta \cdot \underline{\varphi}(\lambda) + (1-\theta)\cdot d$	$\overline{\varphi}$ is d-Lipschitz

What is condition $(*)$?




d
upper bound
$(1-\theta)\dim_L^\theta E$
slope flatter
0
$\lambda\theta$
λ
θ
1

Theorem (Banaji-Chen-R.-Wang)

$$\left\{ \theta \mapsto (1-\theta)\dim_L^\theta E : E \subseteq \mathbb{R}^d \right\}$$

$$=$$

$$\left\{ \varphi : (0,1) \to [0,d] : \begin{array}{l} \varphi(\theta) + \theta\varphi(\lambda) \leqslant \varphi(\lambda\theta) \\ \qquad \leqslant \theta \cdot \varphi(\lambda) + (1-\theta) \cdot d \\ \text{for } \lambda, \theta \in (0,1) \end{array} \right\}$$

Theorem (Banaji-Chen-R.-Wang)

$$\left\{\theta \mapsto (1-\theta)\dim_L^{\theta} E \;:\; E \subseteq \mathbb{R}^d\right\}$$

$$=$$

$$\left\{\varphi : (0,1) \to [0,d] \;:\; \begin{array}{l} \varphi(\theta) + \theta\varphi(\lambda) \leq \varphi(\lambda\theta) \\ (*) \leq \theta \cdot \varphi(\lambda) + (1-\theta)\cdot d \\ \text{for } \lambda, \theta \in (0,1) \end{array}\right\}$$

Note: if E uniform, $(*)$ is replaced with d-Lipschitz

(1) Joint classification?

$$\{\theta \mapsto (\dim_L^\theta E, \dim_A^\theta E) : \emptyset \neq E \subseteq \mathbb{R}^d\}$$

(2) Modified lower spectrum? $\dim_{ML}^\theta E = \sup\{\dim_L^\theta F : F \subseteq E\}$

$$\{\theta \mapsto \dim_{ML}^\theta E : \emptyset \neq E \subseteq \mathbb{R}^d\}$$

(3) Product sets?

$$\left\{\theta \mapsto (\dim_A^\theta E, \dim_A^\theta F, \dim_A^\theta E \times F) : \begin{array}{l} \emptyset \neq E \subseteq \mathbb{R}^d \\ \emptyset \neq F \subseteq \mathbb{R}^k \end{array}\right\}$$

(4) Branching functions in point-line space?

upper 2-scale branching fn

① $\beta(u,v)$ decr in v

② $\beta(u,v) \leq \beta(u,w) + \beta(w,v)$

③ β is d-Lipschitz in u,v

Assouad spectrum

① $\overline{\varphi}$ decr

② $\overline{\varphi}(\lambda\theta) \leq \overline{\varphi}(\theta) + \theta\,\overline{\varphi}(\lambda)$

③ φ is d-Lipschitz

lower 2-scale branching fn

[① $\gamma(u,v)$ decr in v]

$\Uparrow$

② $\gamma(u,v) \geq \gamma(u,w) + \gamma(w,v)$

③ γ is d-Lipschitz in u only

lower spectrum

[① $\underline{\varphi}$ decr]

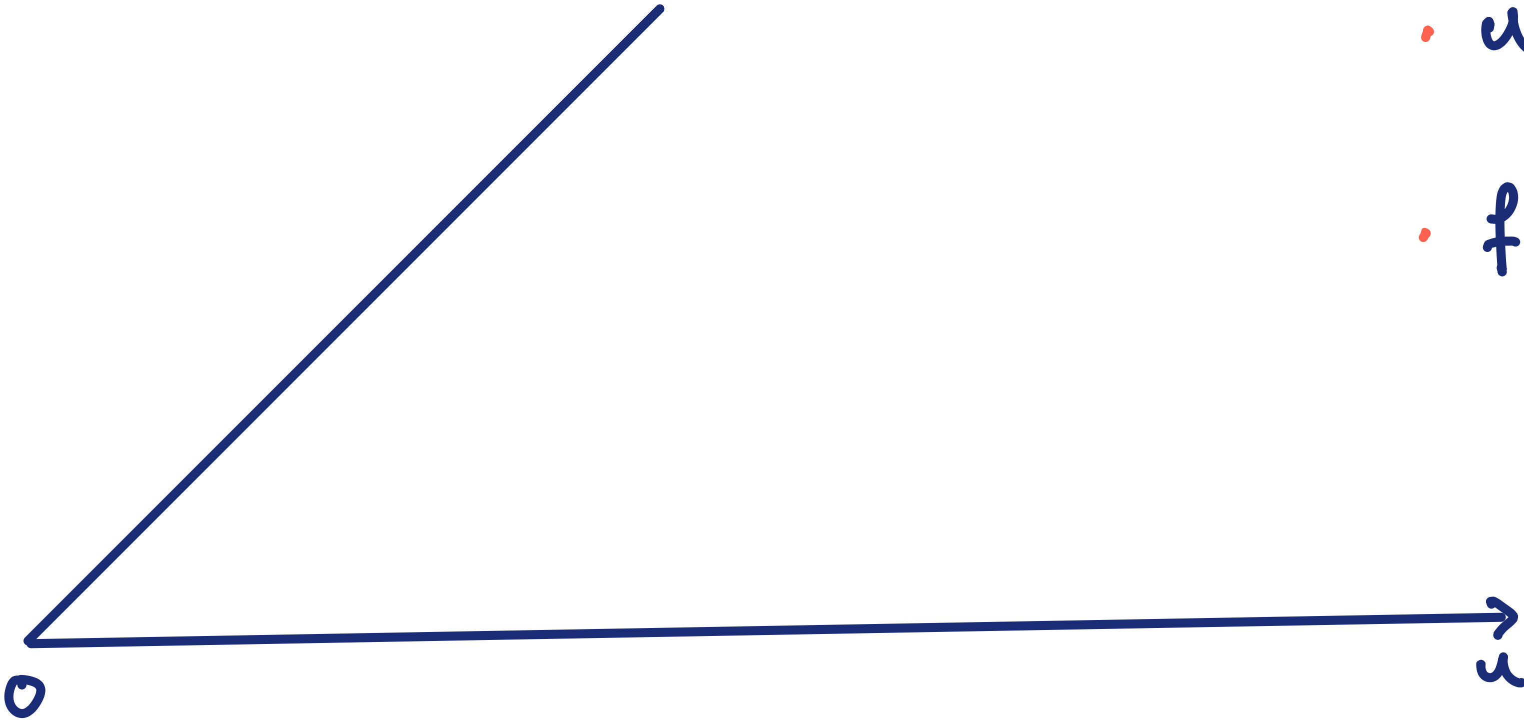
$\Uparrow$

② $\underline{\varphi}(\lambda\theta) \geq \underline{\varphi}(\theta) + \theta\,\underline{\varphi}(\lambda)$

③ $\underline{\varphi}(\lambda\theta) \leq \theta \cdot \underline{\varphi}(\lambda) + (1-\theta)d$

Pf sketch of construction. $f \in \mathcal{L}(d)$ fixed:

- $u \mapsto f(u,v)$ incr, d-Lip
- $f(u,v) \geq f(u,w) + f(w,v)$



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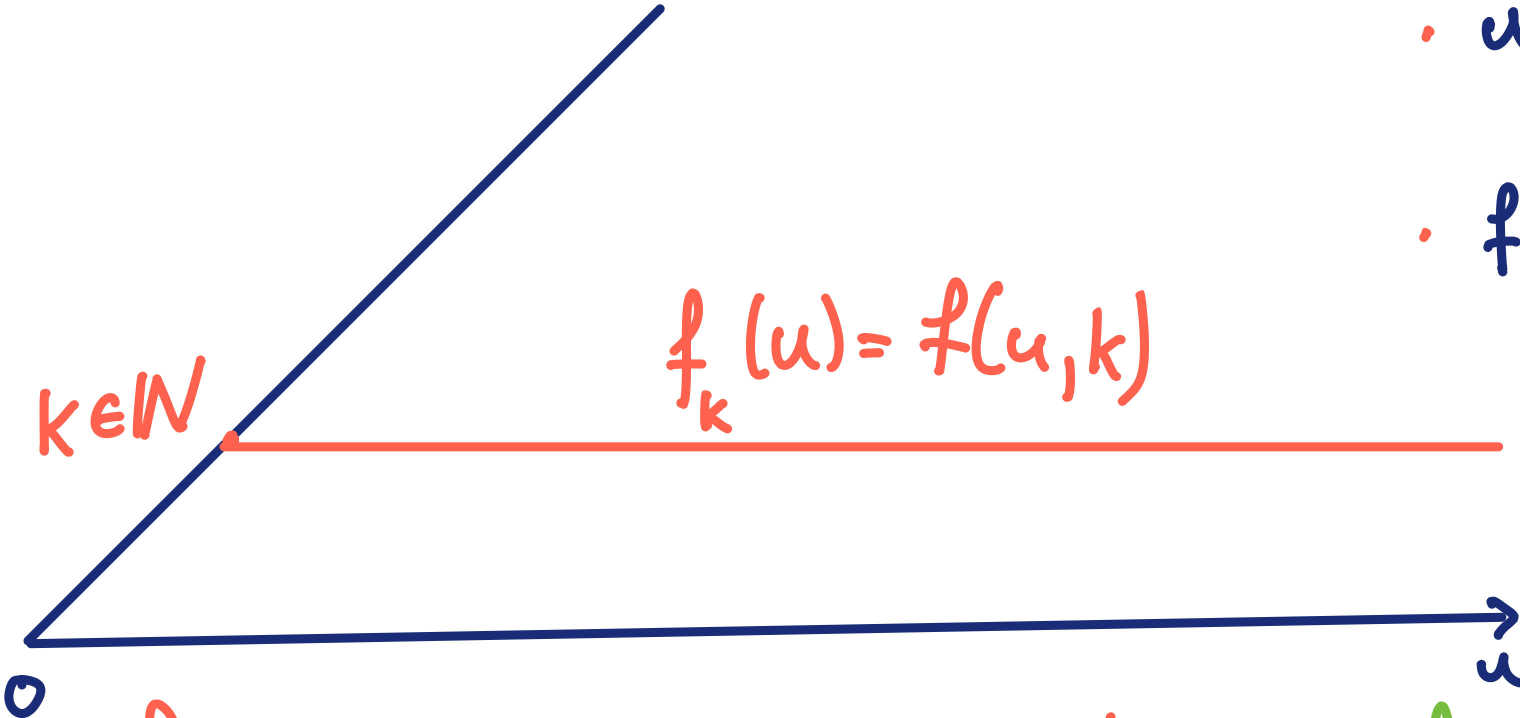

$f_k \rightsquigarrow E_k \subseteq [0, 2^{-k}]$, uniform, branching function f_k

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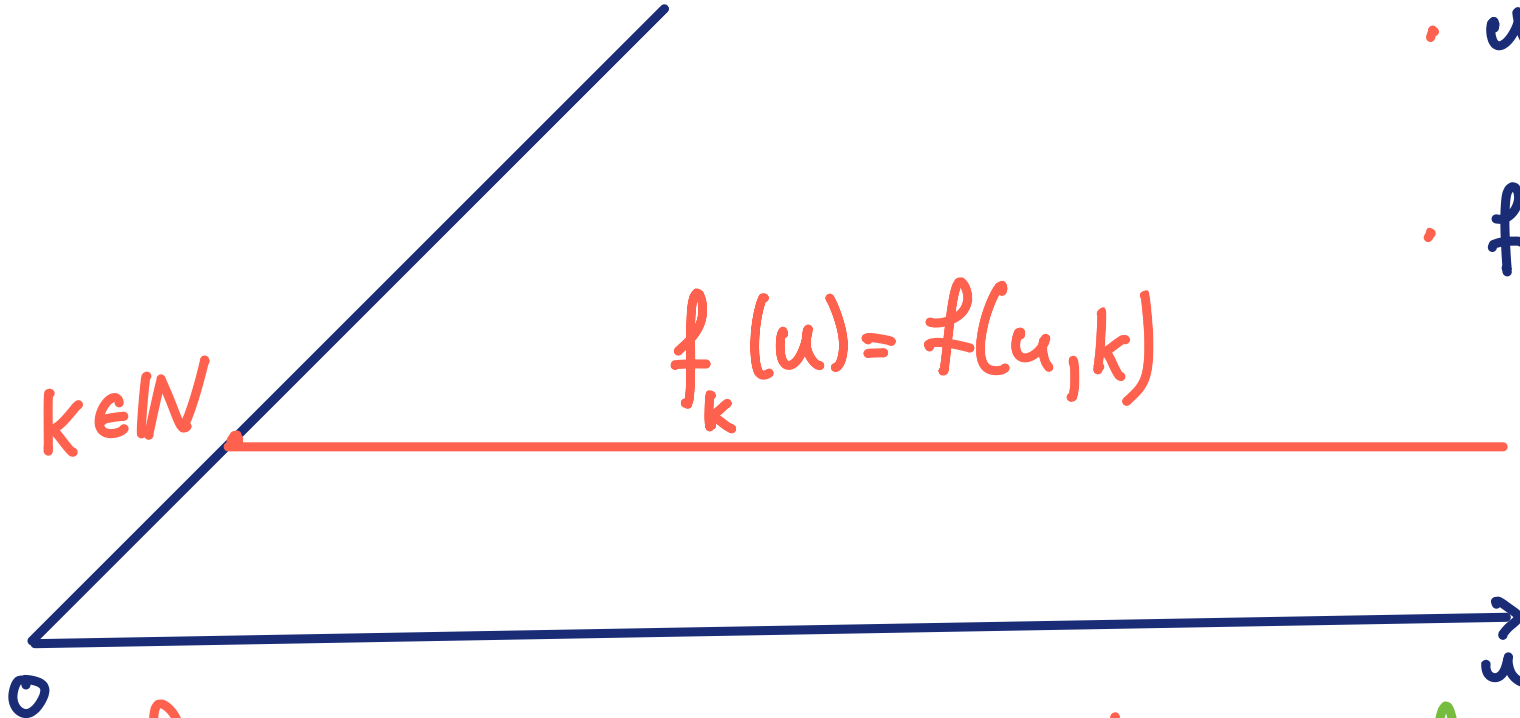

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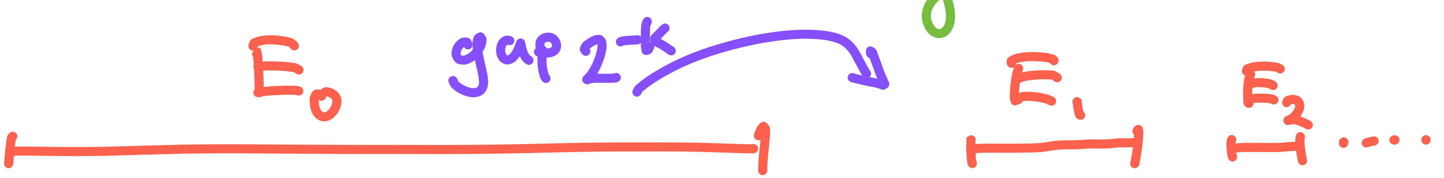

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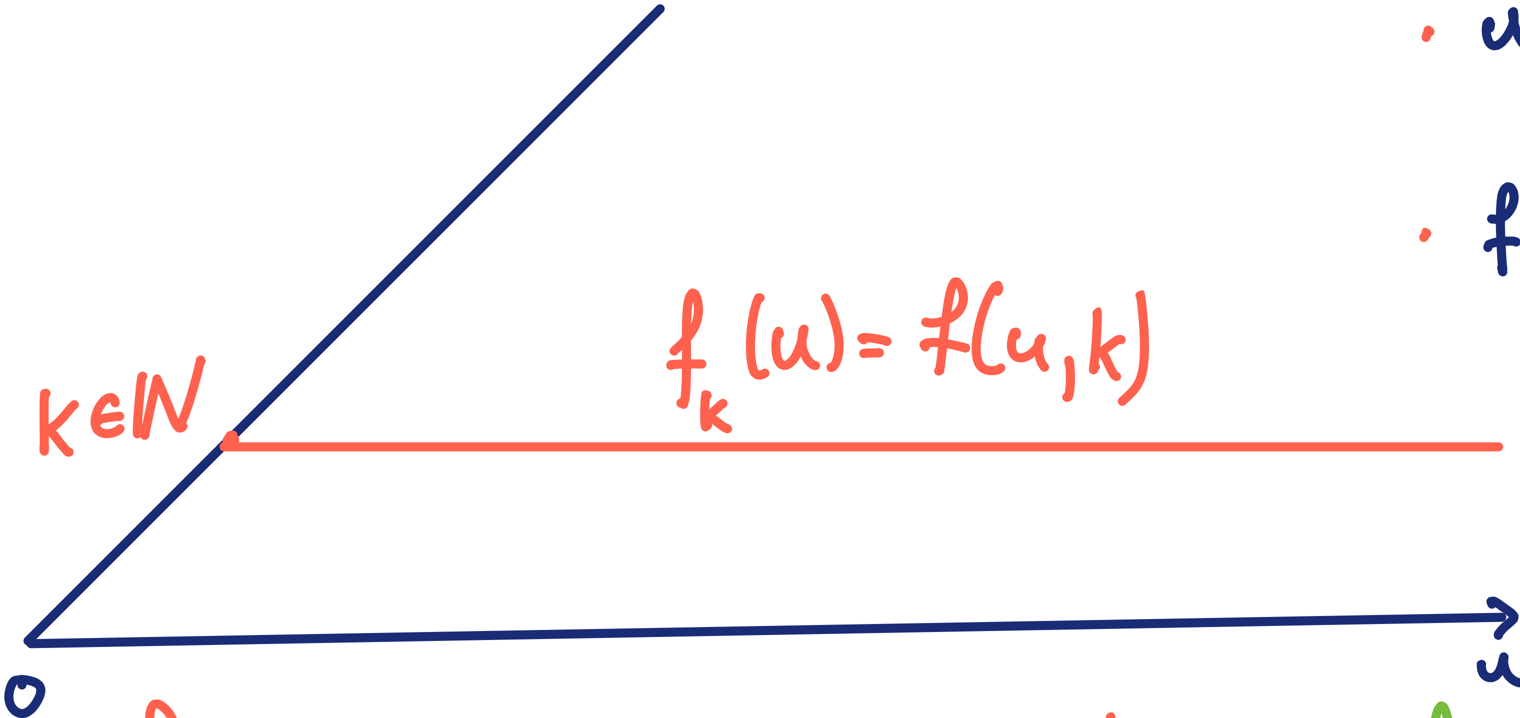

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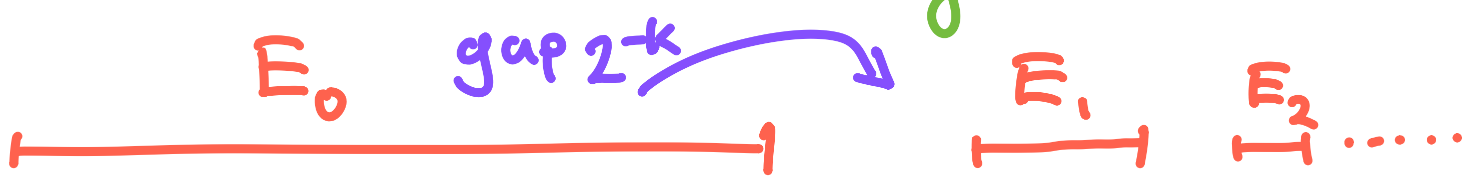
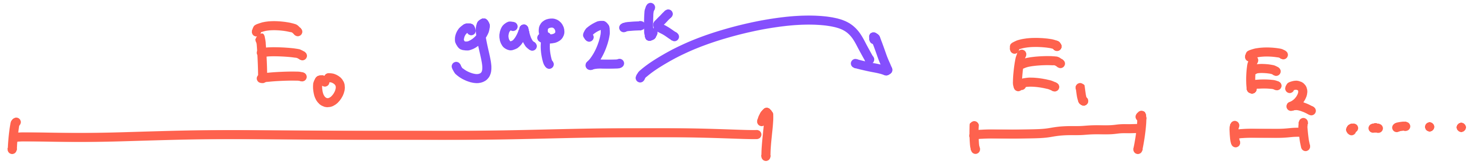


If $x_k \in E_k$: $B(x_k, 2^{-k}) \cap E = E_k$

$\Longrightarrow \gamma_E(u,k) \leqslant f_k(u) = f(u,k)$



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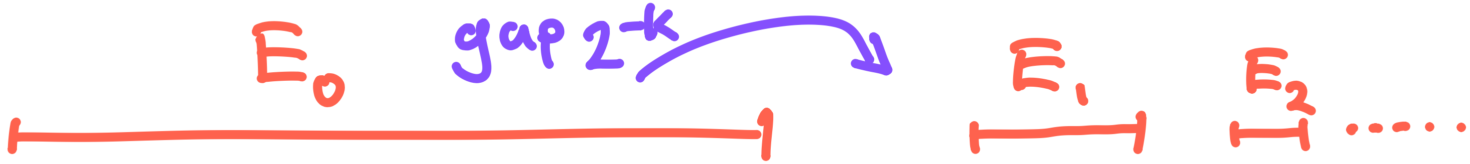
Lower bound?



If $x_k \in E_k$: $B(x_k, 2^{-k}) \cap E = E_k$

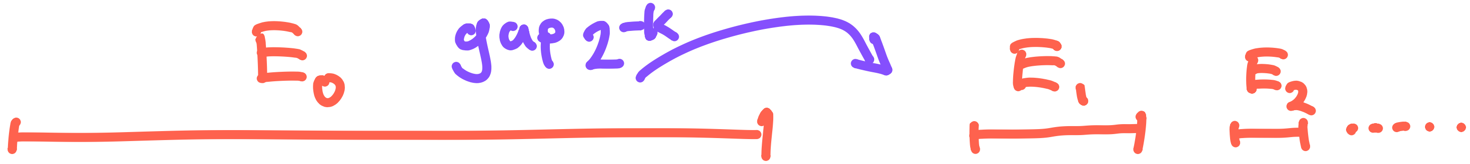
$\Longrightarrow \gamma_E(u,k) \leq f_k(u) = f(u,k)$

$\Longrightarrow \gamma_E(u,v) \leq f(u,v)$

Lower bound?

Suppose $x \in E_j$ where $j \leqslant k$. Then

$$
\begin{aligned}
N_{2^{-u}}(B(x,2^{-k}) \cap E) &= N_{2^{-u}}(B(x,2^{-k}) \cap E_j) \\
&= f_j(u) - f_j(k) && [E_j \text{ uniform}] \\
&= f(u,j) - f(k,j) && [\text{def'n}] \\
&\geqslant f(u,k) && [\text{superadd.}]
\end{aligned}
$$



If $x_k \in E_k$: $\quad B(x_k, 2^{-k}) \cap E = E_k$

$$\implies \gamma_E(u,k) \leq f_k(u) = f(u,k)$$

$$\implies \gamma_E(u,v) \leq f(u,v)$$

Lower bound?

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&= f_j(u) - f_j(k) && [E_j \text{ uniform}] \\
&= f(u,j) - f(k,j) && [\text{def'n}] \\
&\geq f(u,k) && [\text{superadd.}]
\end{aligned}
$$

Otherwise, $E_k \subseteq B(x, 2^{-k})$ [geom. seq. of diameters]

$$
\begin{aligned}
\implies N_{2^{-u}}(B(x, 2^{-k}) \cap E) &\geq N_{2^{-u}}(B(x, 2^{-k}) \cap E_k) \\
&= f(u,k)
\end{aligned}
$$