Conformal dimension beyond self-similarity

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I: Conformal Assouad dimension

Quasisymmetric maps.

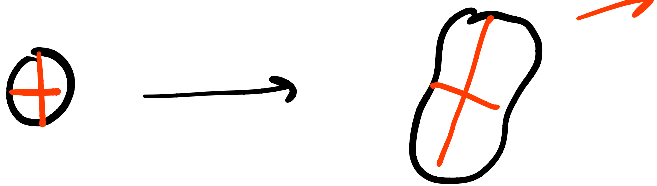
Let $f: X \to Y$ be a map b/w metric spaces;

$\eta: [0,\infty) \to [0,\infty)$ a homeomorphism.

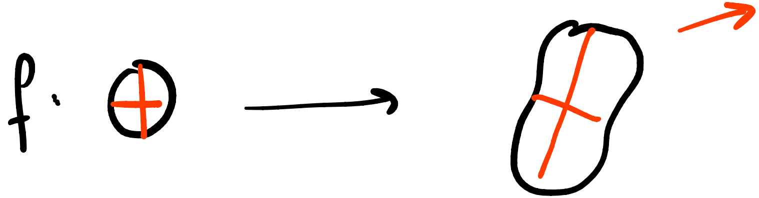
We say f is η-quasisymmetric if

$$\frac{d_Y(f(x), f(y))}{d_Y(f(x), f(z))} \leq \eta\left(\frac{d_X(x,y)}{d_X(x,z)}\right)$$

$f : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is quasisymmetric (q.s.) iff f is quasiconformal; i.e. f "preserves infinitesimal balls"

f :  max. eccentricity H.

In Euclidean space, f is quasisymmetric iff f is quasiconformal; i.e. f "preserves infinitesimal balls"



max. eccentricity H.

Examples:

- bi-Lipschitz maps
- "snowflaking": $(X,d) \to d^{\varepsilon}(x,y) = d(x,y)^{\varepsilon}$ $(\varepsilon \in (0,1))$

  $id : (X,d) \to (X,d^{\varepsilon})$

The usual geometry questions.

- Classify spaces up to quasisymmetric equivalence

- Improve the geometry of a space w/ quasisymmetric map:

Doubling and Assouad dimension.

(X,d) is **doubling** if $\exists M > 0$ s.t.,
any ball $B(x,2r)$ can be covered by
M balls of radius r.

Doubling property is preserved by q.s.

Theorem (Assouad) (X,d) is doubling iff X is q.s. equivalent to a subset of $\mathbb{R}^m$ for some $m \in \mathbb{N}$

$$(X,d) \xrightarrow{\text{q.s.}} (X,d^{\varepsilon}) \overset{\text{bi-Lip.}}{\longleftrightarrow} \mathbb{R}^m$$

Assouad dimension (Def 1 of 3)

Motivation: quantify doubling

$= \dim_B X \qquad \dim_M$

$$\dim_A X = \inf\Big\{ \alpha \geq 0 : \exists C>0 \ \forall\, 0<r\leq R<1 \ \forall x\in K \quad N_r(X\cap B(x,R)) \leq C\cdot\Big(\frac{R}{r}\Big)^{\alpha} \Big\}$$

N_r = # balls radius r required to cover ..

Fact: X doubling iff $\dim_A X < \infty$

Assouad dimension (Def 2 of 3)

Motivation: measure obstruction to Ahlfors regularity

$$\dim_A X = \inf \left\{ \dim_H Y : \begin{array}{l} \cdot\ Y \text{ AD regular,} \\ \cdot\ X \text{ bi-Lip equiv to a subset of } Y \end{array} \right\}$$

[Recall: X AD s-regular if $\mathcal{H}^s(B(x,r)) \approx r^s \quad \forall x, r$

Exercise: If X AD regular then

$$\dim_H X = \dim_A X.$$

Assouad dimension (Def 3 of 3)

Motivation: dimension "at infinity"

Assouad dimension (Def 3 of 3)

(For simplicity work in $\mathbb{R}^m$)

$K \subset \mathbb{R}^m$ compact: $\lambda(K - x) \cap B(0,1)$

"magnification"

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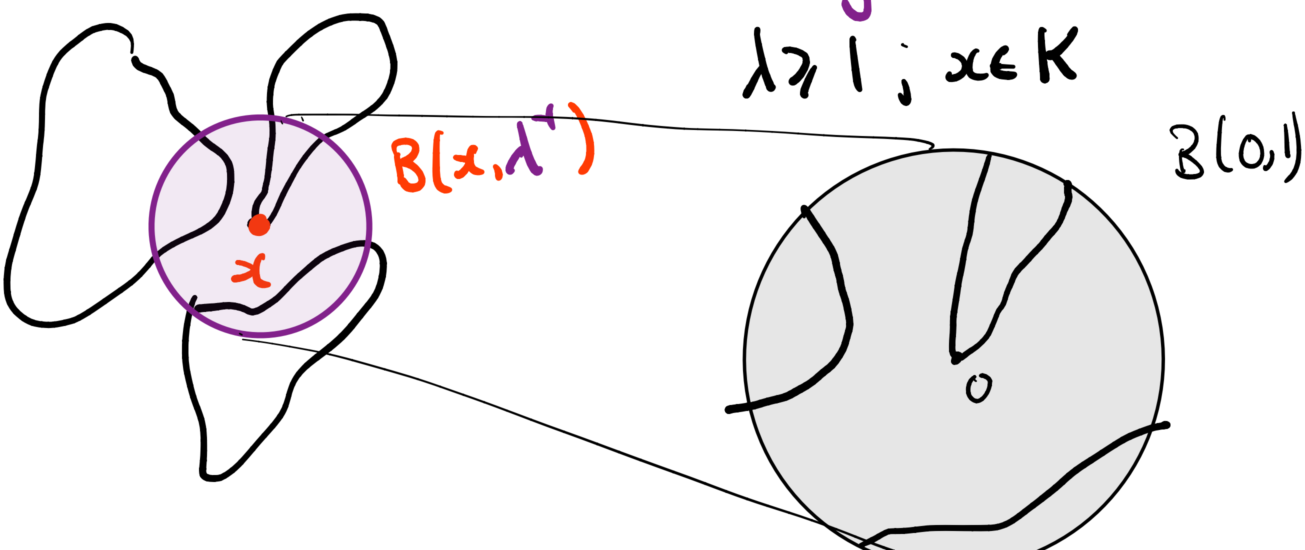
$K \subset \mathbb{R}^m$ compact : $\lambda(K - x) \cap B(0,1)$

"magnification"

$\lambda \geq 1$; $x \in K$



Assouad dimension (Def 3 of 3)

A weak tangent is a compact set $F \subset B(0,1)$ s.t.

$$F = \lim_{n \to \infty} \lambda_n (K - x_n)$$

Hausdorff metric

with

- $x_n \in K$
- $\lambda_n \longrightarrow \infty$

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space of weak tangents
$\shortparallel$

$$\dim_A K = \max \{ \dim_H F : F \in \mathrm{Tan}(K) \}$$

Equivalence of (1) and (2) is not so difficult; equivalence w/ (3) is quite a bit deeper (essentially due to Furstenberg from 60s, but explicit connection made by Käenmäki-Ojala-Rossi 17')

Will return to this later!

Conformal dimension

$$\mathrm{Cdim}_A X = \inf\{\dim_A Y : \ \exists \text{ q.s. } f : X \to Y\}$$

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Why conformal Assouad dimension?

- invariant for quasisymmetric maps

$\exists\, c>0$ s.t. $B(x,r)\setminus B(x,cr) \neq \emptyset$ for all $x \in X$, r small

- if X complete + uniformly perfect

then

$$\mathrm{Cdim}_A X = \inf\left\{ s : \begin{array}{l} Y \text{ q.s. equiv to } X \\ Y \text{ AD } s\text{-regular} \end{array} \right\}$$

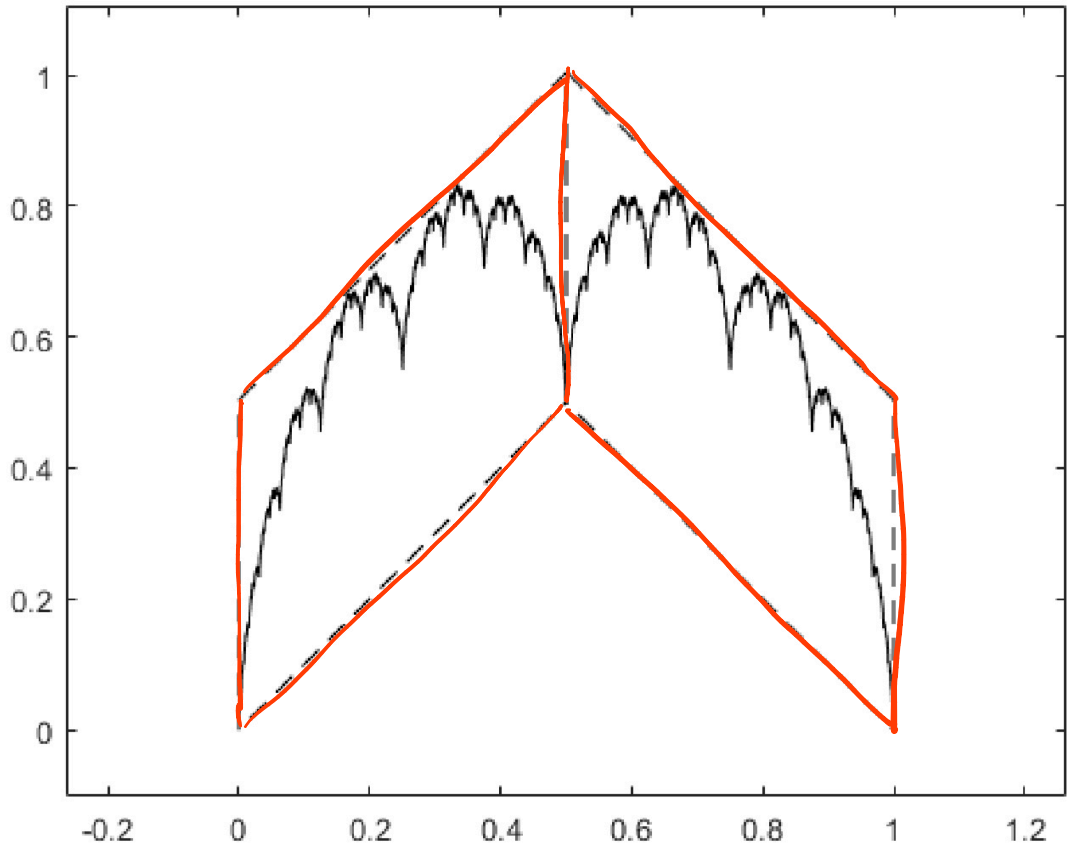
II : Self-affine sets

Finite set of maps $T_i(x) = A_i x + t_i$

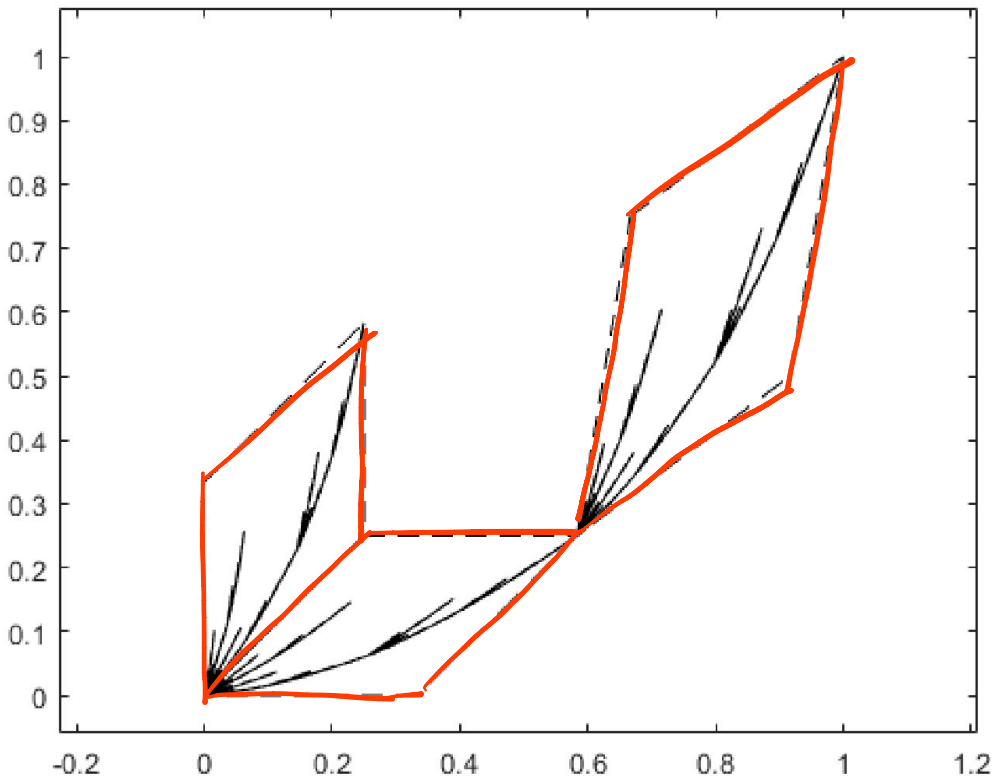
w/ $\|A_i\| < 1$

Fact: $\exists$ unique K s.t. $K = \bigcup_i T_i(K)$

Assume: $T_i : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$



1
0.8
0.6
0.4
0.2
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2


1
0.9
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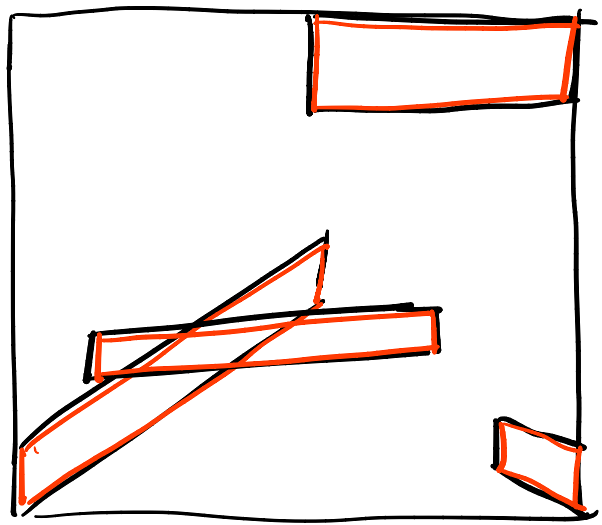
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.2
0
0.2
0.4
0.6
0.8
1
1.2

Could be much worse...



Many fundamental questions about self-affine sets are open:

- "Canonical" formulas for dimensions of K; independent of translations?
- regularity questions (e.g. do upper / lower box dims coincide?)
- projections, slicing, ...

Matrix geometry assumptions:

(Recall $T_i(x) = A_i x + t_i$)

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- $(A_i)_i$ dominated if there exists invariant multicone $\Delta \subset \mathbb{RP}^1$ [Δ proper subset + non-empty interior + $A_i \Delta \subset \Delta$]

Matrix geometry assumptions:

(Recall $T_i(x) = A_i x + t_i$)

- $(A_i)_i$ dominated if there exists invariant multicone $\Delta \subsetneq \mathbb{RP}^1$ $\left[\begin{array}{l}\Delta \text{ finite union of closed intervals} \\ A_i \Delta \subset \Delta^\circ \quad \forall i\end{array}\right.$

- $(A_i)_i$ irreducible: there is no $V \in \mathbb{RP}^1$ s.t. $A_i V = V$ for all i.

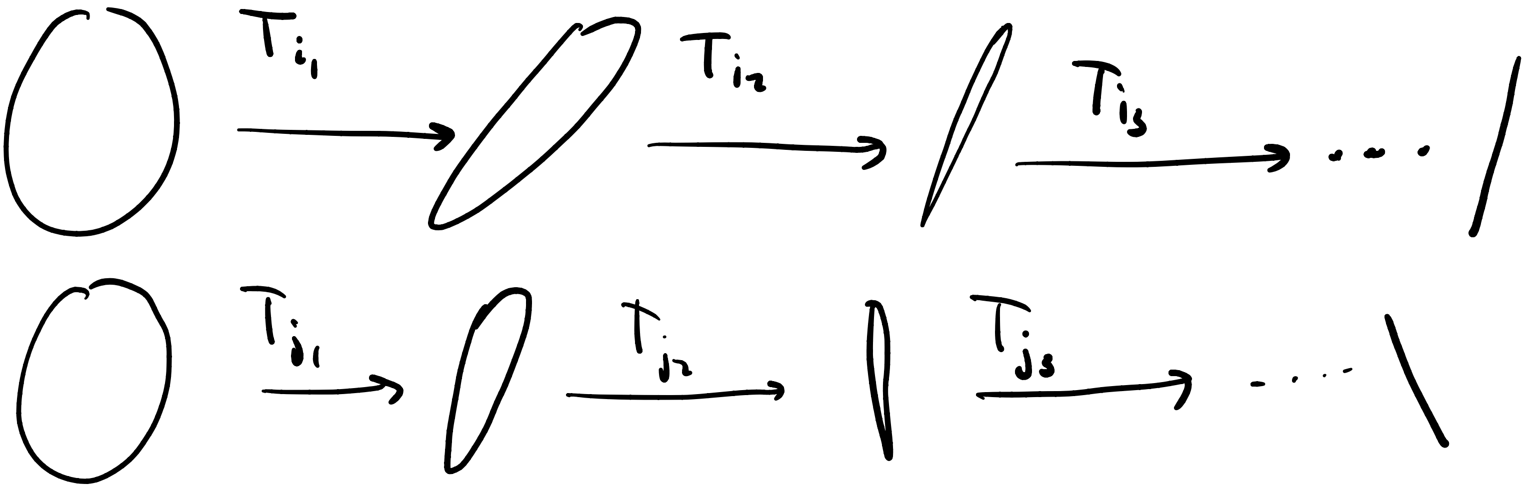
Matrix geometry assumptions:

(Recall $T_i(x) = A_i x + t_i$)

- $(A_i)_i$ dominated if there exists invariant multicone (proper subset $\Delta \subset \mathbb{RP}^1$ s.t. $A_i \Delta \subset \Delta$ for all i)

- $(A_i)_i$ irreducible: there is no $V \in \mathbb{RP}^1$ s.t. $A_i V = V$ for all i.

[domination is open subset of $GL_2(\mathbb{R})^n$
irreducible is full measure ...]



- exponentially fast
- uniformly over $(i_n)_{n=1}^{\infty}$
- "limit direction" may depend on $(i_n)_{n=1}^{\infty}$

Question: what is $\mathrm{Cdim}_A K$?

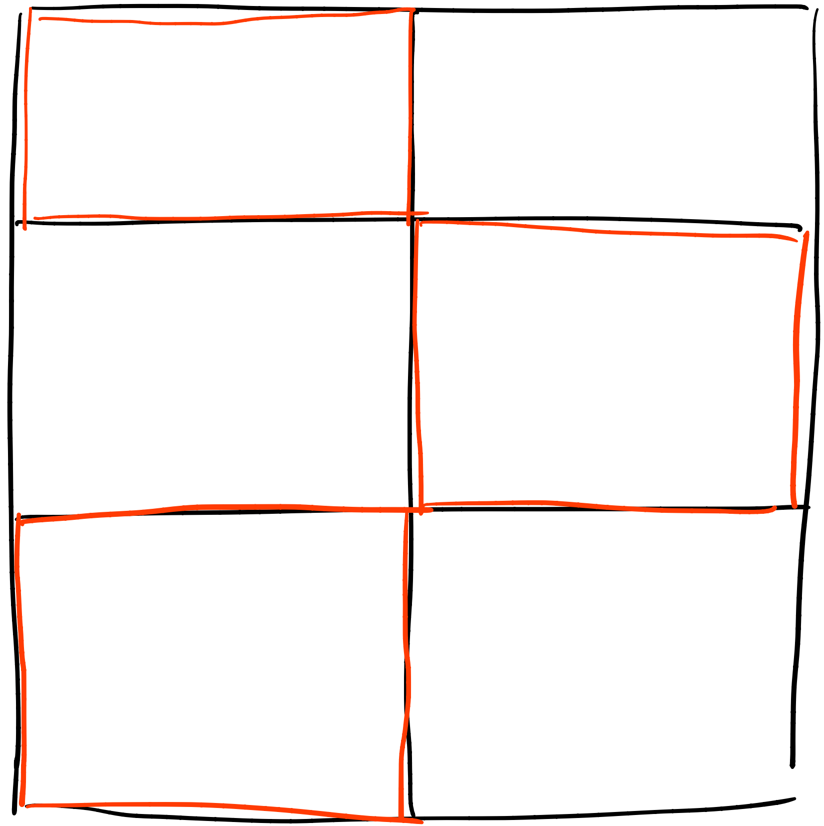
Theorem (Bárány–Käenmäki–Yu; 2021+)

- $\dim_H K \geq 1$
- matrix parts irreducible + dominated
- translations are s.t. $T_i(K) \cap T_j(K) = \emptyset$ for $i \neq j$

then $\dim_A K = \mathrm{Cdim}_A K$

Following Bárány–Käenmäki–Rossi; Ferguson–Fraser–Sahlsten; Fraser–Jordan; Mackay; Käenmäki–Ojala–Rossi; …

Obstructions "at infinity"



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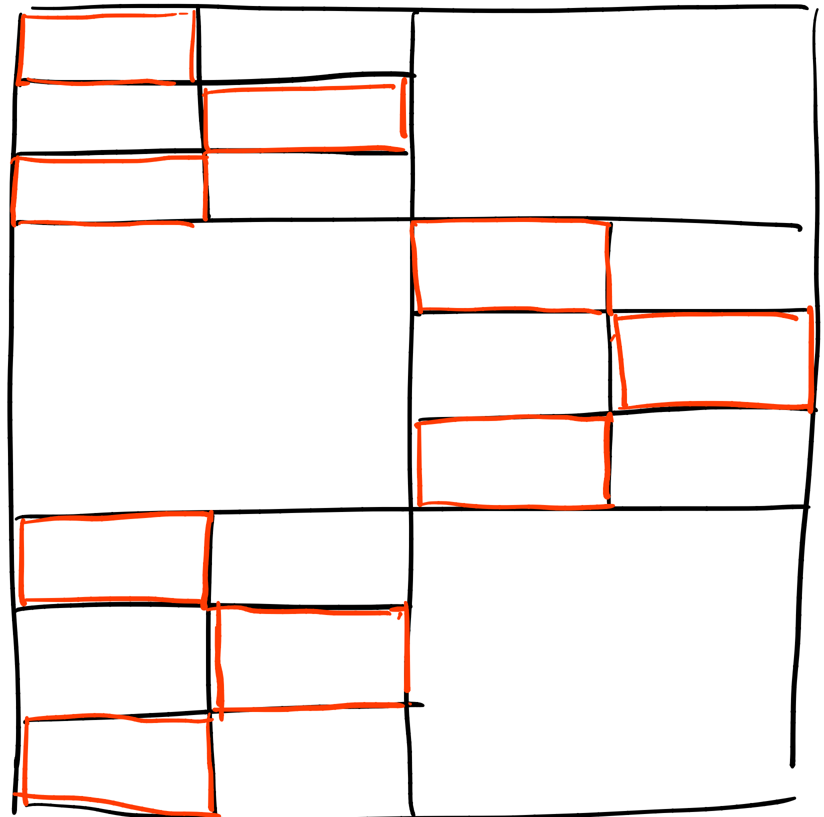
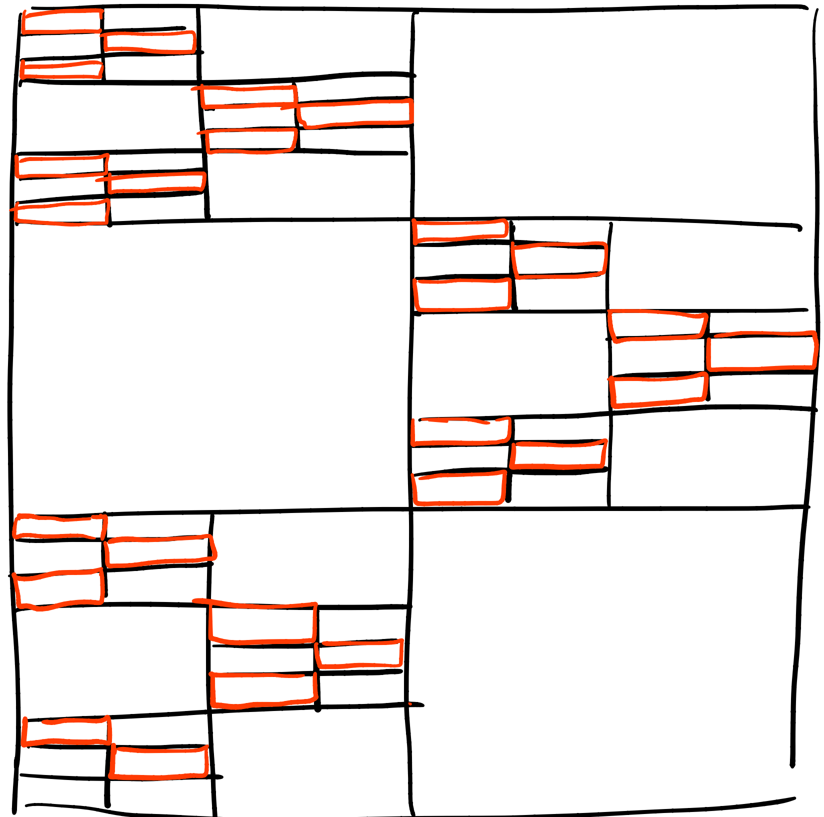


Obstructions "at infinity"



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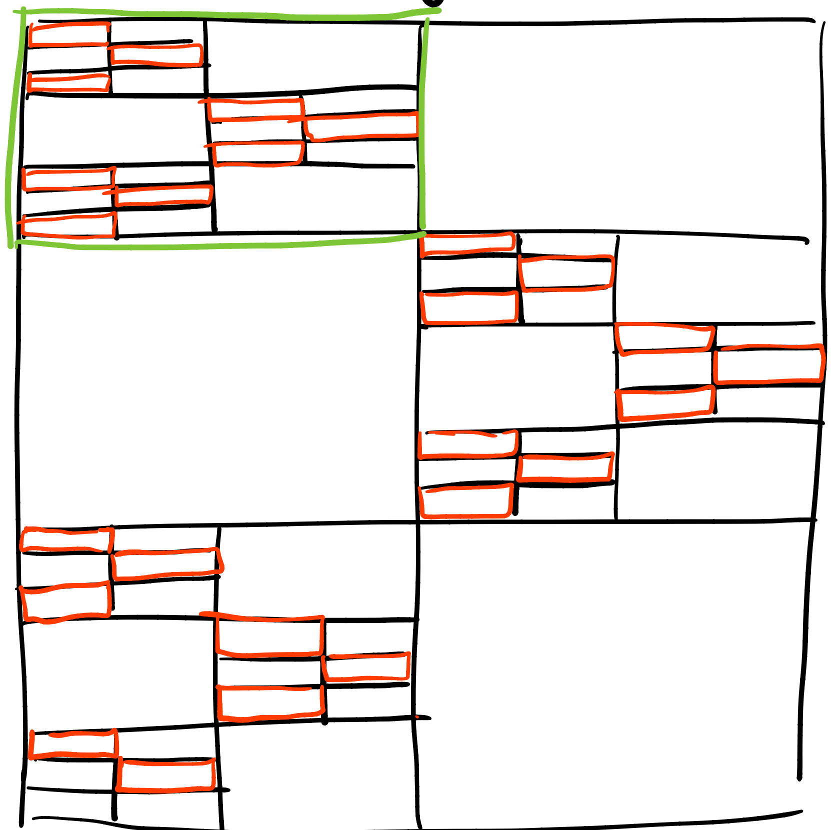

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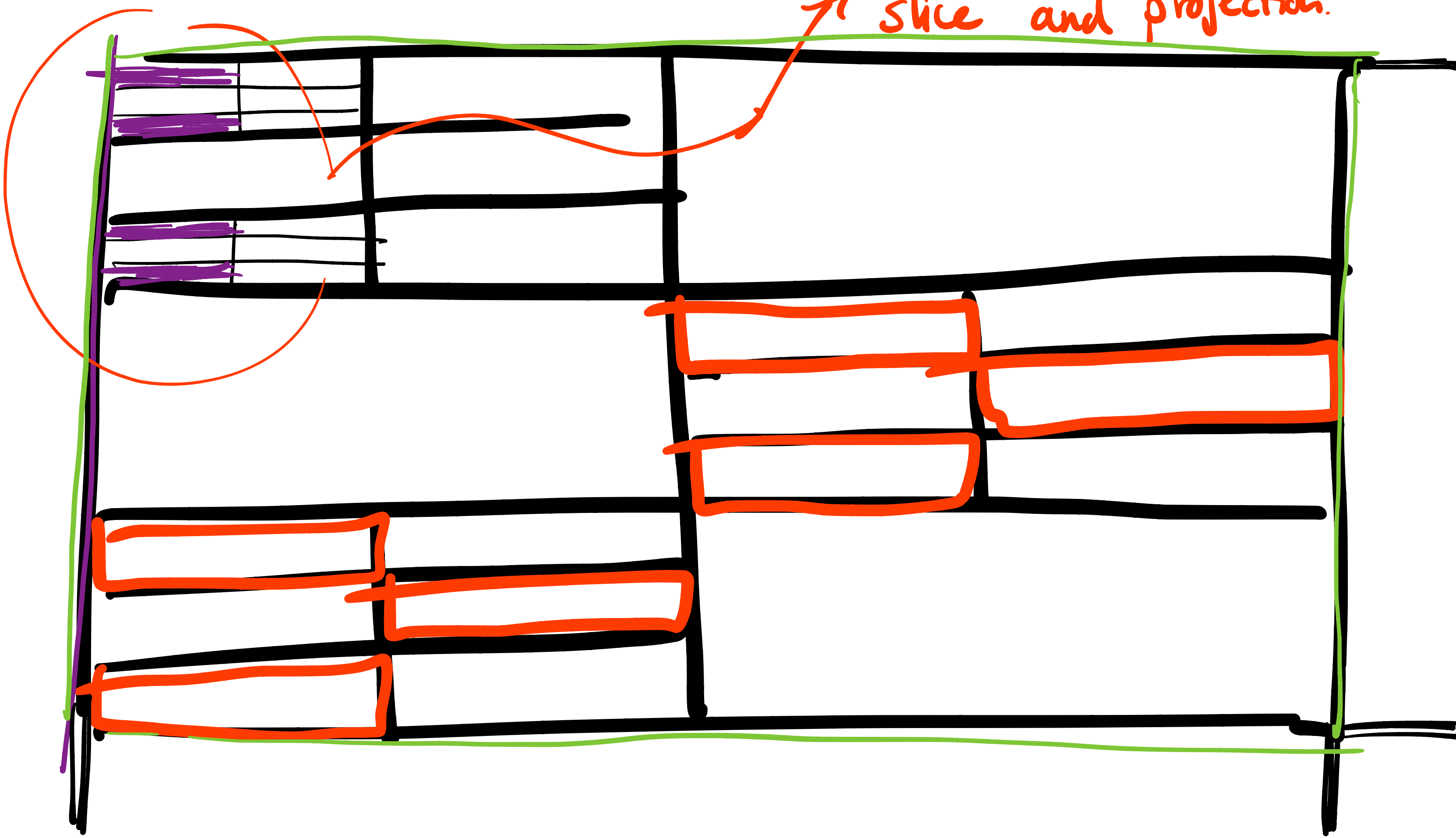


In limit: K has weak tangent of the form

$$[0,1] \times \text{slice}$$

s.t. $\dim_H([0,1] \times \text{slice}) = \dim_A K$.

Moreover, sets with large curve families have minimal conformal dimension.

$$\Longrightarrow \quad C\dim_A K \geq C\dim_H([0,1] \times \text{slice}) = \dim_H([0,1] \times \text{slice})$$
$$= \dim_A K$$

↳ q.s. induces a q.s. on weak tangent

$\therefore$ we reduce the problem to:

$\exists$? weak tangent which is q.s. equivalent to $[0,1] \times E$ for some $E \subset \mathbb{R}$ with $\dim_H E = \dim_A K - 1$

Theorem (Bárány–Käenmäki–Yu; 2021+)

- $\dim_H K \geq 1$
- matrix parts irreducible + dominated
- translations are s.t. $T_i(K) \cap T_j(K) = \emptyset$ for $i \neq j$

Then $\dim_A K = \mathrm{C}\dim_A K$

Pf. sketch of BKY result.

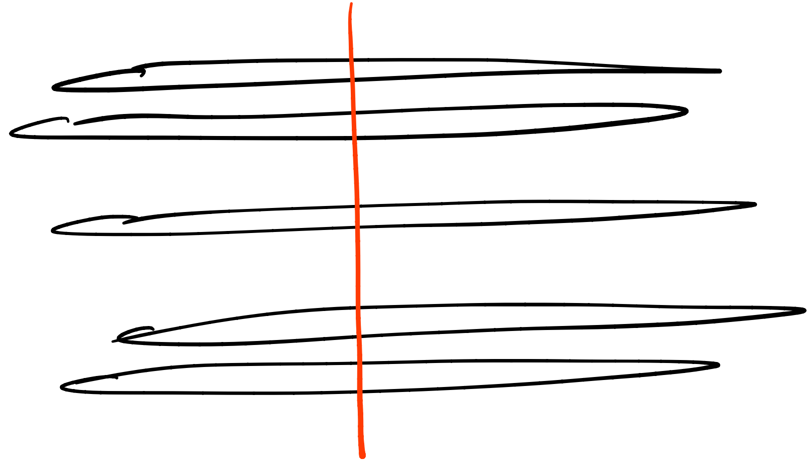
① By deep result of Bárány-Hochman-Rapaport

$\dim_H K \geq 1$ + irreducible $\Rightarrow$ $\dim_H \pi(K) = 1$ for all "directions of max contraction"

② Take weak tangent which realizes Assouad dim.

③ Use domination + separation to obtain product structure

③ :



④ Since $\dim_H \{\text{projection}\} = 1$, use Marstrand slicing theorem to get large slice.

III: Furstenberg's "dynamics on fractals"

Theorem (Bárány – Käenmäki – Yu; 2021+)

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- matrix parts irreducible + dominated
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Then $\dim_A K = C\dim_A K$

Theorem (Anttila – R. , 2024+)

- $\cancel{\dim_H K \geqslant 1}$ → $\dim_A K \geqslant 1$
- matrix parts irreducible + dominated
- ~~translations are s.t.~~ $\cancel{T_i(K) \cap T_j(K) = \emptyset}$
  ~~for~~ $\cancel{i \neq j}$

Then $\dim_A K = \mathrm{C}\dim_A K$

Theorem (Anttila – R. , 2024+)

- $\dim_A K \geq 1$
- matrix parts irreducible + dominated

then $\dim_A K = \mathrm{Cdim}_A K$

- If $\dim_A X < 1$ then $\mathrm{Cdim}_A X = 0$.
- irreducible required; $\exists K$ s.t. $\dim_A K > 1$ but $\mathrm{Cdim}_A K = 0$.
- removing domination seems difficult

Bárány–Hochman–Rapaport requires some assumption on translations, plus $\dim_H K \geq 1$ is required

$\Rightarrow$ need new way to guarantee large projection.

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$\boxed{\text{Theorem (Orponen, 2021) For all cpct } F \subset \mathbb{R}^2, \quad \dim_H \{ e \in S^1 : \dim_A \pi_e(F) < \min\{\dim_A F, 1\} \} = 0}$

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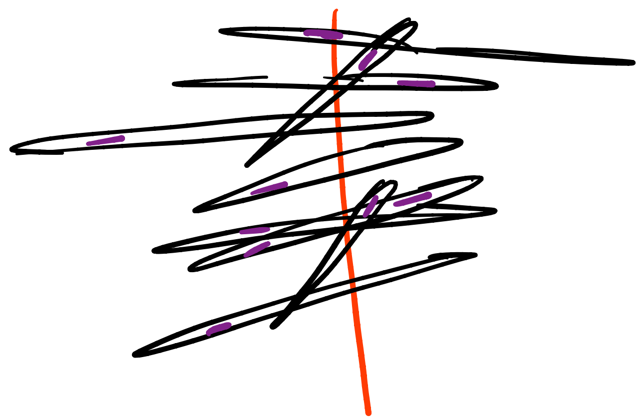
"Strong Marstrand projection" for Assouad dimension

$\Rightarrow$ If $\dim_A K \geq 1$ + irreducible $\Rightarrow$ $\dim_A \pi(K) = 1$ for a "good" π.

Still many difficulties:

- $\dim_A \pi(K)$ much harder to exploit (no Marstrand slicing theorem)
- without separation, "small-scale" geometry is much worse.



How to improve this to a strong configuration?

Lemma ("Furstenberg amplification")

Suppose

- K t-dimensional at scale r
- $0 < s < t$; $\rho > 0$; r small$(t-s, \rho)$

Then $\exists$ new scale δ, and $B(x, \delta)$

s.t. $B(x, \delta) \cap K$ is large at all scales $\in (\rho \cdot \delta, \delta)$.

Proof sketch

1. **Lemma (Weak dimension conservation)**
   Let $F \subset \mathbb{R}^2$ be non-empty + compact. Then
   $\exists\, E \in \mathrm{Tan}(F)$, $x \in \pi(E)$

   $$\underbrace{\dim_H \pi(x)^{-1} \cap E}_{\text{slice of weak tangent}} + \dim_A \pi(F) \geq \dim_A F$$

Proof sketch

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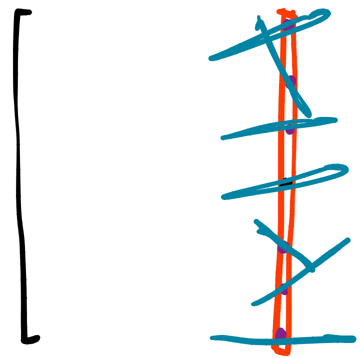
Correct numerology but missing product structure

② Pullback

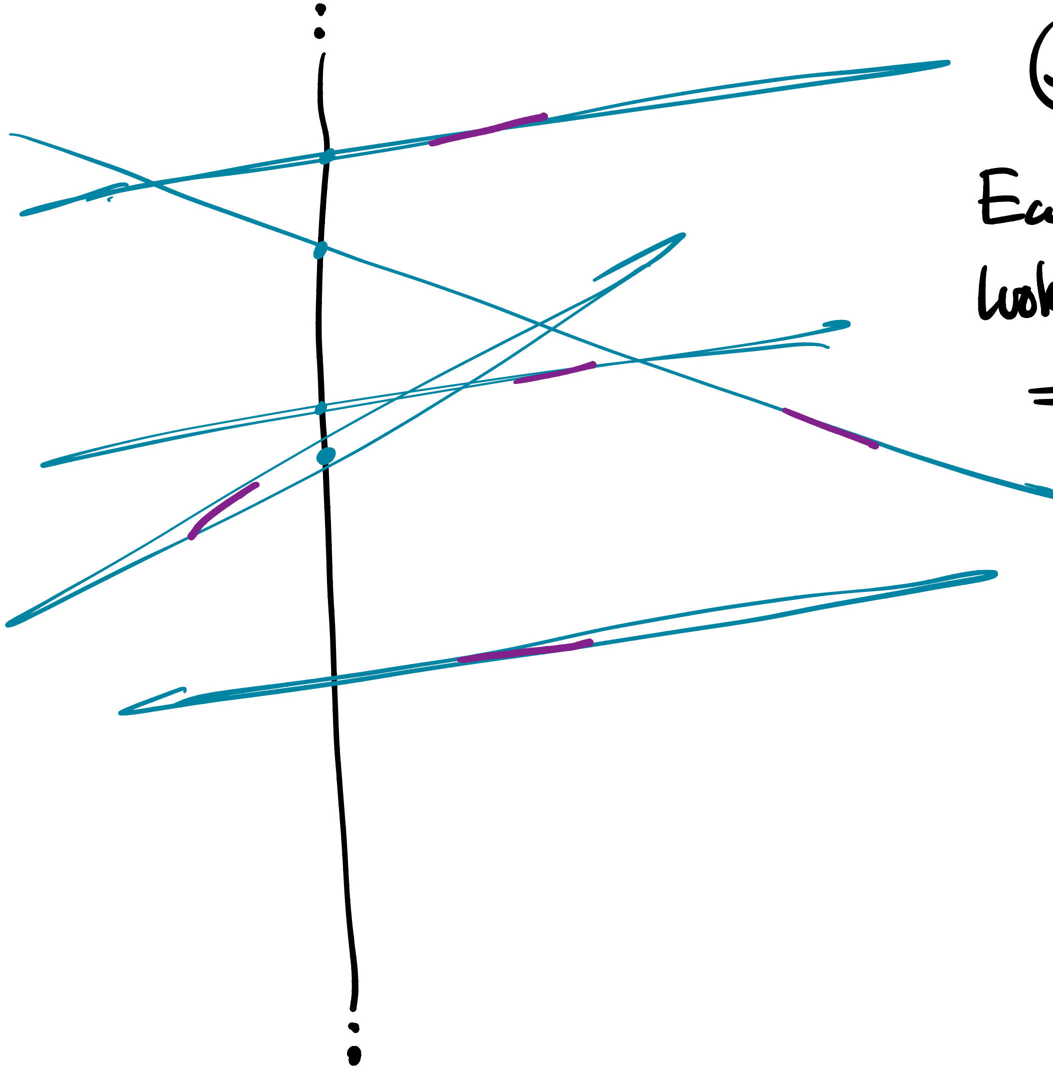
slice of weak tan becomes tube

$r \ll 1$



large dimension

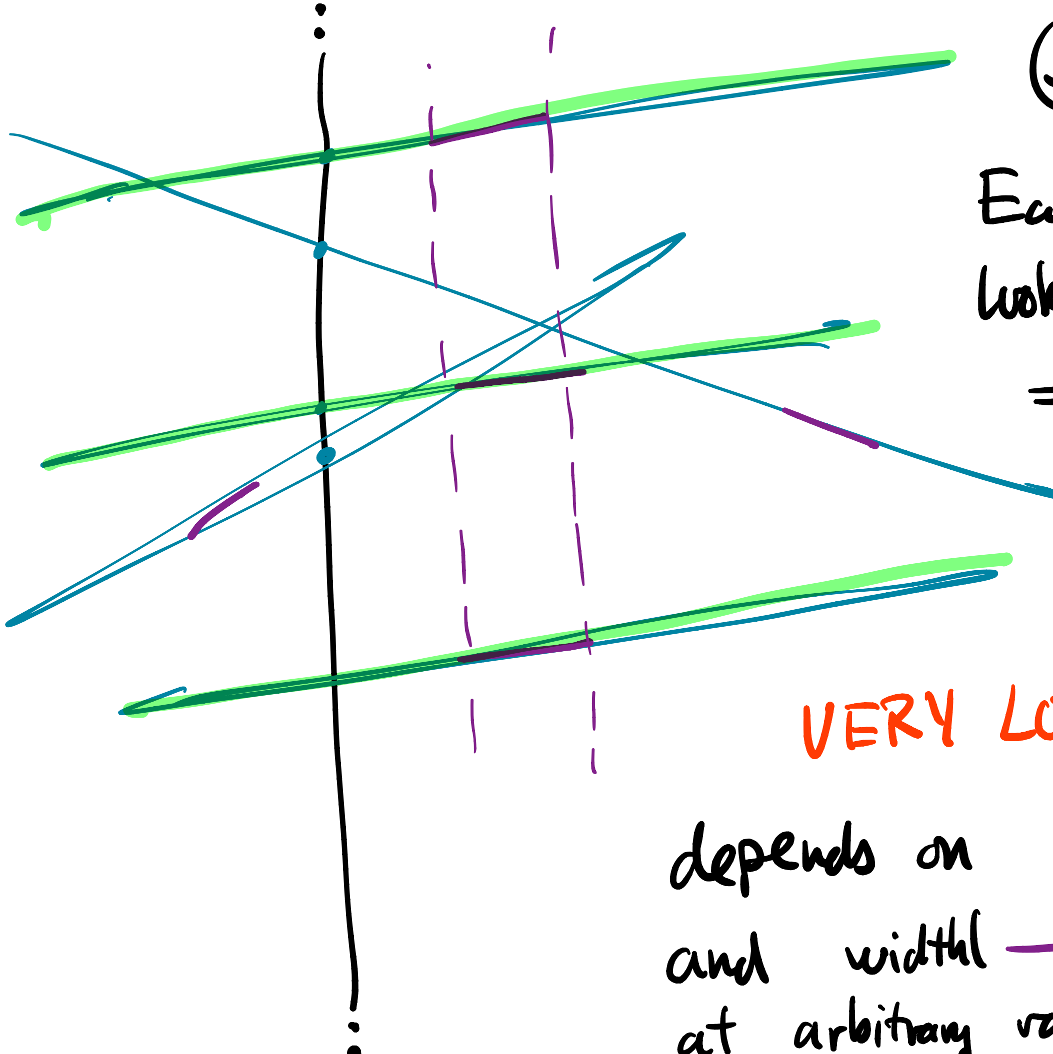
$\rightsquigarrow$ attach cylinder $T_i(K)$ to each point



③ Pigeonhole

Each looks like $\pi(K)$

$\Rightarrow$ find which "realizes Assouad dim"



③ Pigeonhole

Each  looks like $\pi(K)$

$\Rightarrow$ find — which "realizes Assouad dim"

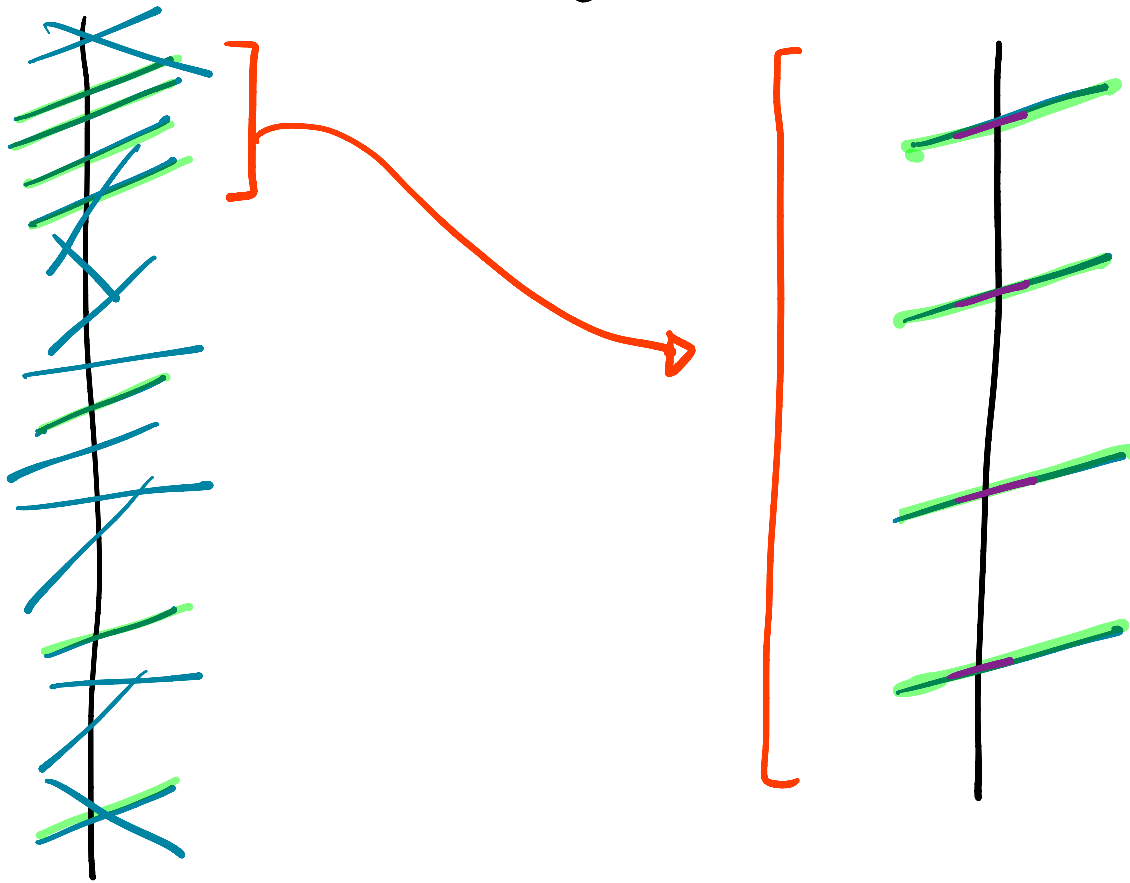
VERY LOSSY :

depends on size of —

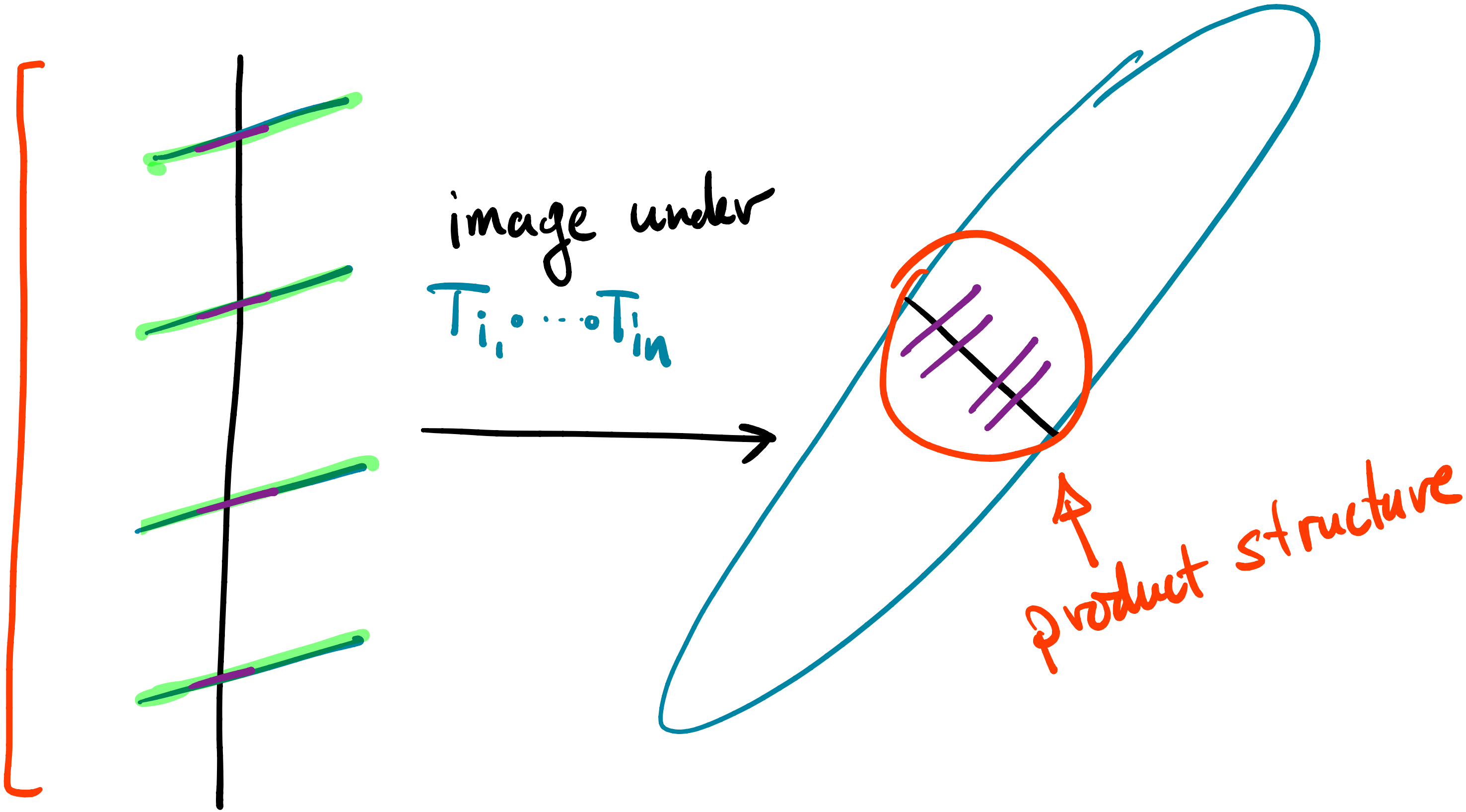
and width(—) $\to 0$

at arbitrary rate

④ Amplify configuration



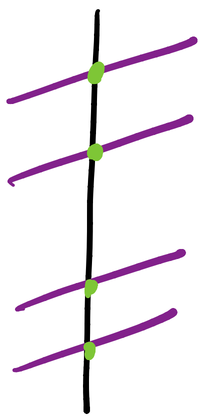
⑤ Use domination + self-affinity



⑥ Limit.

Repeat above procedure along sequence of

- — $\to$ max. weak tangent of $\pi(K)$.
- slices $\to$ large slice from dimension conservation lemma



$$\dim_H(\text{—}) = \dim_A \pi(K)$$

$$+$$

$$\dim_H(\,|\,) = \dim_H \pi^{-1}(x) \cap E$$

$$\geq$$

$$\dim_A K$$