Box dimensions of countably-generated self-conformal sets

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Conformal Dynamics.

- $f \in C^{1+\alpha}$ is conformal if the Jacobian $Df(x)$ is a multiple of a similarity ("locally angle preserving").

- f is uniformly expanding if $\exists\ \lambda > 1$ $\|Df^n(x)u\| \geq \lambda^n \|u\|$

Theorem (Falconer '86, Barreira '96, Gatzouras–Peres '97)
Suppose f is conformal + uniformly expanding. If Λ is compact and satisfies $f(\Lambda)=\Lambda$, then

$$\dim_H \Lambda = \overline{\dim}_B \Lambda .$$

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Conformality is essential (this fails e.g. for sets invariant under affine maps;

- Bedford '84
- McMullen '84

$\dim_H \Lambda < \dim_B \Lambda$

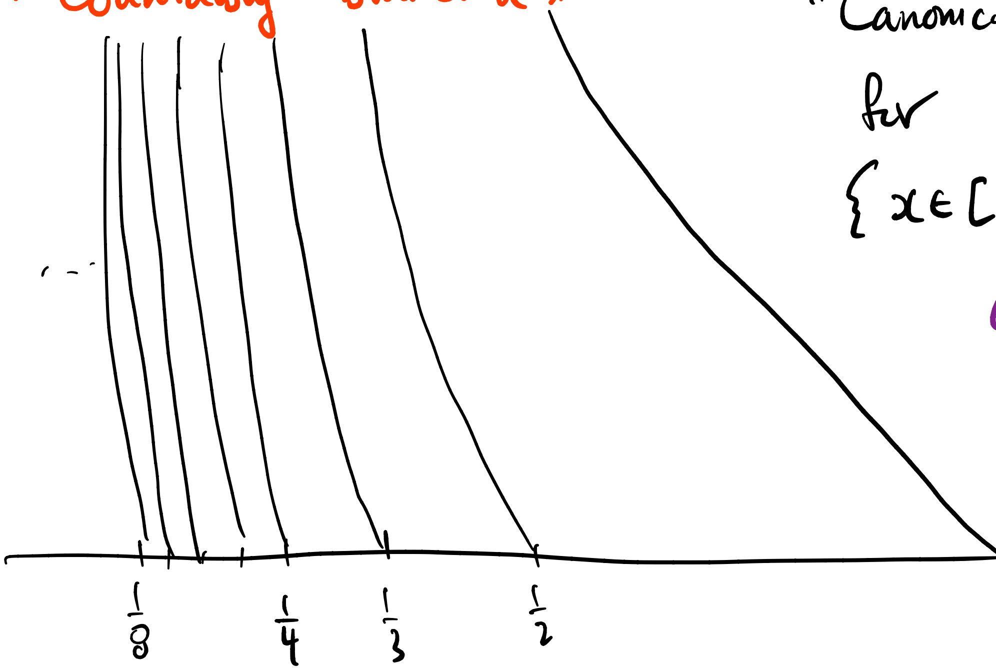
- Jurga '23 $\dim_H \Lambda < \underline{\dim}_B \Lambda < \overline{\dim}_B \Lambda$

Non-compactness?

$$f(x) = \frac{1}{x} \pmod 1$$

$$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \qquad x = [0; a_1, a_2, a_3 \ldots]$$

* countably branched *



"Canonical invariant sets":

for $I \subset \mathbb{N}$,

$$\{x \in [0; a_1 a_2 a_3 \ldots] : a_i \in I \ \forall i\}$$

continued fraction missing digit set

More generally,

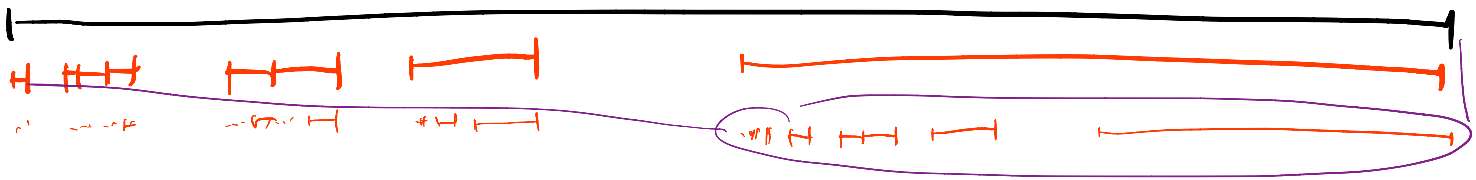
$$\Lambda = \bigcup_{i \in \mathcal{I}} f_i(\Lambda)$$ *Not compact (in general)*

for conformal IFS $\{f_i\}_{i \in \mathcal{I}}$ in $\mathbb{R}^d$.

(Framework of Mauldin–Urbański. '96)

"dynamical" $\approx$ "open set condition"

Structure of Λ



- Λ contains undistorted, rescaled copies of itself.
- Λ contains orbit sets $\{f_i(0) : i \in \mathcal{I}\}$ at all small scales.

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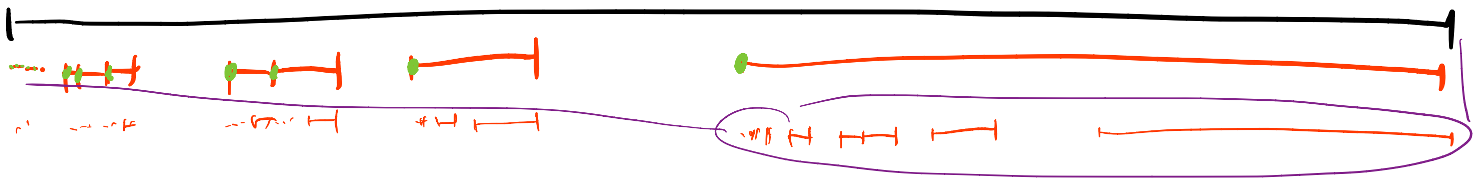

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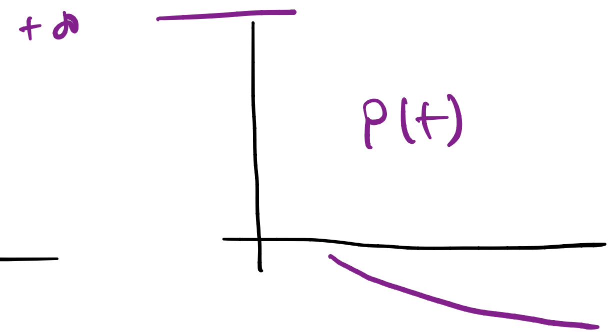
— Notation — set $F = \{f_i(0) : i \in \mathcal{I}\}$

Pressure

$$P(t) = \lim_{n \to \infty} \frac{1}{n} \log \sum_{I \in \mathcal{I}^n} \| f_I' \|$$



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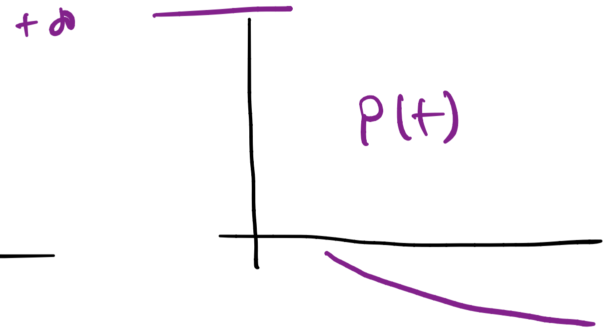

Theorem (Mauldin–Urbański '96 '99)

- $h := \dim_H \Lambda = \inf\{t \geq 0 : P(t) < 0\}$
- $\dim_P \Lambda = \overline{\dim}_B \Lambda = \max\{h, \overline{\dim}_B F\}$

Questions

- Does $\dim_B \Lambda$ exist?
- If not, what can be said about $\underline{\dim}_B \Lambda$?
- Does $\underline{\dim}_B \Lambda$ depend only on $(h, \underline{\dim}_B F, \overline{\dim}_B F)$?

Easy bounds (following Mauldin–Urbański '99)

$$(*)\quad \max\{h, \underline{\dim}_B F\} \leq \underline{\dim}_B \Lambda \leq \max\{h, \overline{\dim}_B F\}$$

$(*)$ Provides **two** mechanisms for $\dim_B \Lambda$ to exist:

- $\dim_B F$ exists
- F is small ; i.e. $\overline{\dim}_B F \leq h$.

Theorem (Banaji – R. '24+)

(1) $\dim_B \Lambda$ exists $\iff$ $\max\{h, \underline{\dim}_B F\} = \max\{h, \overline{\dim}_B F\}$

Theorem (Banaji–R. '24+)

(1) $\dim_B \Lambda$ exists $\Longleftrightarrow$ $\max\{h, \underline{\dim}_B F\} = \max\{h, \overline{\dim}_B F\}$

(2) Sharp bounds in terms of $(h, \underline{\dim}_B F, \overline{\dim}_B F, d)$

d = ambient dimension

if $\overline{\dim}_B F > h$: (otherwise, see (1))

$$\max\{h, \underline{\dim}_B F\} \leq \underline{\dim}_B \Lambda \leq h + \frac{(\overline{\dim}_B F - h)\cdot(d-h)\cdot \underline{\dim}_B F}{d \cdot \overline{\dim}_B F - h \cdot \underline{\dim}_B F}$$

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(3) Any configuration permitted by (1)/(2) is possible.

(i.e. $\underline{\dim}_B \Lambda$ not a function of $(h, \underline{\dim}_B F, \overline{\dim}_B F)$).

Corollary: $\exists$ restricted digit sets for continued fractions s.t. $\dim_H \Lambda < \underline{\dim}_B \Lambda < \overline{\dim}_B \Lambda$.

Non-compactness of invariant set is essential. Note that f is still conformal + uniformly expanding.

Def'n. Given $E \subseteq \mathbb{R}^d$ bounded and $0 < r < 1$,

$$s_E(r) = \frac{\log N_r(E)}{\log(1/r)}.$$ "box dimension at scale r"

Theorem. Set

- $\psi(r,\theta) = (1-\theta)\cdot h + \theta\, s_F(r^\theta)$
- $\psi(r) = \sup_{\theta \in (0,1)} \psi(r,\theta)$

Then

$$\lim_{r \to 0} \left[s_\Lambda(r) - \psi(r) \right] = 0$$

What does this formula mean heuristically?

Recall: (1) Λ contains undistorted, rescaled copies of itself.

(2) Λ contains orbit set F

(2) $\Rightarrow$ $$N_r(F) \leq N_r(\Lambda)$$

(1) $\Rightarrow$ Λ is "h-dimensional" between all pairs of scales and "most" locations:

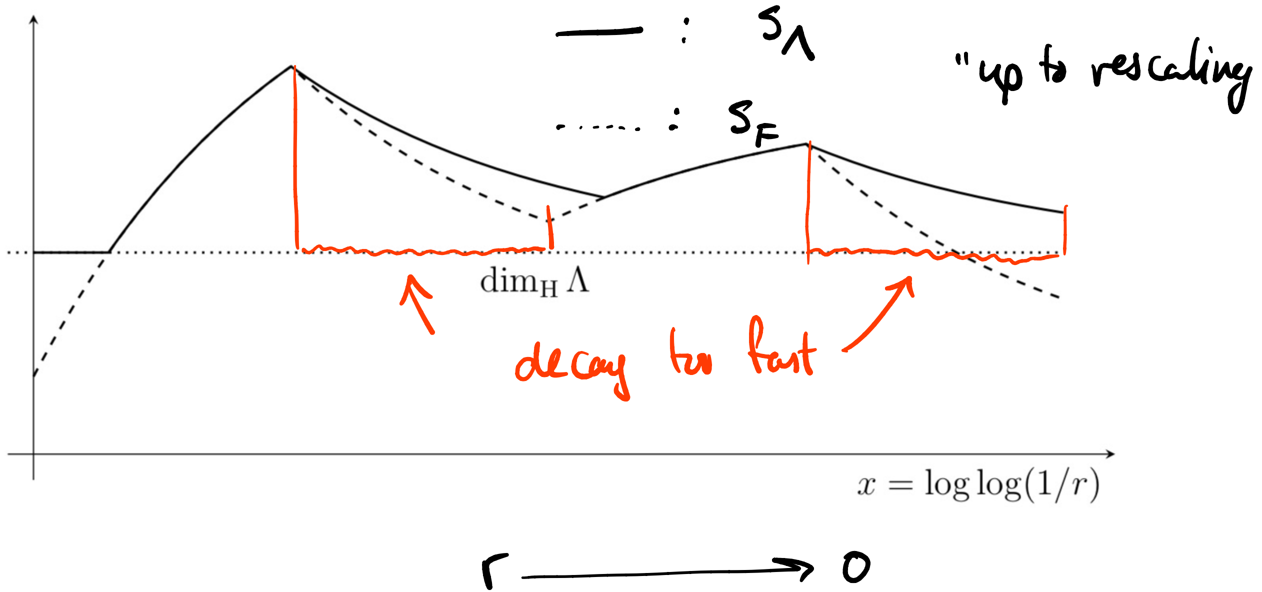
$$N_r(\Lambda \cap B(x,R)) \gtrsim \left(\frac{R}{r}\right)^h$$

(1) $\mathcal{N}_r(F) \leqslant \mathcal{N}_r(\Lambda)$

(2) $\mathcal{N}_r(\Lambda \cap B(x,R)) \gtrsim \left(\frac{R}{r}\right)^h$

Condition (2) is a growth rate condition: $\mathcal{N}_r(\Lambda)$ cannot grow too slowly.

Theorem. $\mathcal{N}_r(\Lambda)$ is as small as possible while satisfying (1) and (2).


: s_Λ
: s_F
"up to rescaling"
$\dim_H \Lambda$
decay too fast
$x = \log\log(1/r)$
$r \longrightarrow 0$

The central idea of the upper bound

$$N_r(\Lambda) = N_r\Big(\bigcup_{i\in I} f_i(\Lambda)\Big)$$

$$\leq N_r\Big(\bigcup_{\substack{i\in I\\ r_i\leq r}} f_i(\Lambda)\Big) + N_r\Big(\bigcup_{\substack{i\in I\\ r_i > r}} f_i(\Lambda)\Big)$$

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if $r_i \leq r$, then
$f_i(\Lambda) \subseteq$ r-nbhd of F.

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$$\leq N_r(F) + \sum_{\substack{i \in \mathcal{I} \\ r_i > r}} N_r(f_i(\Lambda))$$

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$$= N_r(F) + \sum_{\substack{i\in\mathcal{I}\\ r_i> r}} N_{r\cdot r_i^{-1}}(\Lambda)$$

$$N_r(\Lambda) \leqslant N_r(F) + \sum_{\substack{i \in \mathcal{I} \\ r_i > r}} N_{r r_i^{-1}}(\Lambda) \qquad (*)$$

Iterate $(*)$ m times:

$$N_r(\Lambda) \leqslant \sum_{\substack{\underline{i} \in \mathcal{I}^k \\ 0 \leq k < m \\ r_{\underline{i}} > r}} N_{r r_{\underline{i}}^{-1}}(F) + \sum_{\substack{\underline{i} \in \mathcal{I}^m \\ r_{\underline{i}} > r}} N_{r r_{\underline{i}}^{-1}}(\Lambda)$$

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$$\left\{ \begin{array}{c} \underline{i} \in \mathcal{I}^m \\ r_{\underline{i}} > r \end{array} \right\} = \emptyset$$ for all m sufficiently large

BUT f_i are uniformly contracting:

$$N_r(\Lambda) \leqslant \sum_{\substack{i \in \mathcal{I}^* \\ r_i > r}} N_{r r_i^{-1}}(F).$$

$$N_r(\Lambda) \leqslant \sum_{\substack{\mathbf{i}\in\mathcal{I}^* \\ r_{\mathbf{i}} > r}} N_{r r_{\mathbf{i}}^{-1}}(F).$$

Let $\theta_{\mathbf{i}}$ be such that

$$r r_{\mathbf{i}}^{-1} = r^{\theta_{\mathbf{i}}}$$

$$N_{r r_{\mathbf{i}}^{-1}}(F) = r_{\mathbf{i}}^{h} \left(\frac{1}{r}\right)^{(1-\theta_{\mathbf{i}})\cdot h + \theta_{\mathbf{i}}\, s_F(r^{\theta_{\mathbf{i}}})} \leqslant r_{\mathbf{i}}^{h} \left(\frac{1}{r}\right)^{\psi(r)}$$

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THEREFORE

$$N_r(\Lambda) \leqslant \left(\frac{1}{r}\right)^{\psi(r)} \sum_{\substack{i \in \mathcal{I}^* \\ r_i > r}} r_i^h$$

h is critical exponent s.t.

$$< 1.$$
$$\approx$$

- This is essentially the full proof of the (easier) upper bound in self-similar case.

- Lower bound: show every step is sharp (much more work!)

- Conformal case: take initial high iteration; smoothing estimates; ...

- Main theorem uses asymptotic formula
  + geometric / dimensional properties
  + constructions