Classifying Dimension Spectra

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Dimension Spectra?

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e.g. Hausdorff + Box

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define natural parametrized family of dimensions.

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e.g. Hausdorff + Box (intermediate dimensions)

Falconer + Fraser + Kempton, 2020

e.g. Box + Assouad (Assouad spectrum)

Fraser + Yu, 2018

define natural parametrized family of dimensions

Intermediate Dimensions

$$\overline{\dim}_B K = \inf \Big\{ s \geq 0 : \text{for all } \delta \text{ suff. small, } \exists \text{ cover } \{U_i\} \text{ of } K \text{ s.t.}$$

$$\cdot \ \operatorname{diam} U_i = \delta$$

$$\cdot \ \sum_i (\operatorname{diam} U_i)^s \leq 1 \Big\}$$

Intermediate Dimensions

$\overline{\dim}_\theta K = \inf \{ s \geq 0 :$ for all δ suff. small, $\exists$ cover $\{U_i\}$ of K s.t.

- $\delta^{\frac{1}{\theta}} \leq \operatorname{diam} U_i \leq \delta$
- $\sum_i (\operatorname{diam} U_i)^s \leq 1 \}$

[for $\theta \in (0,1)$]

(note: also "lower" version)

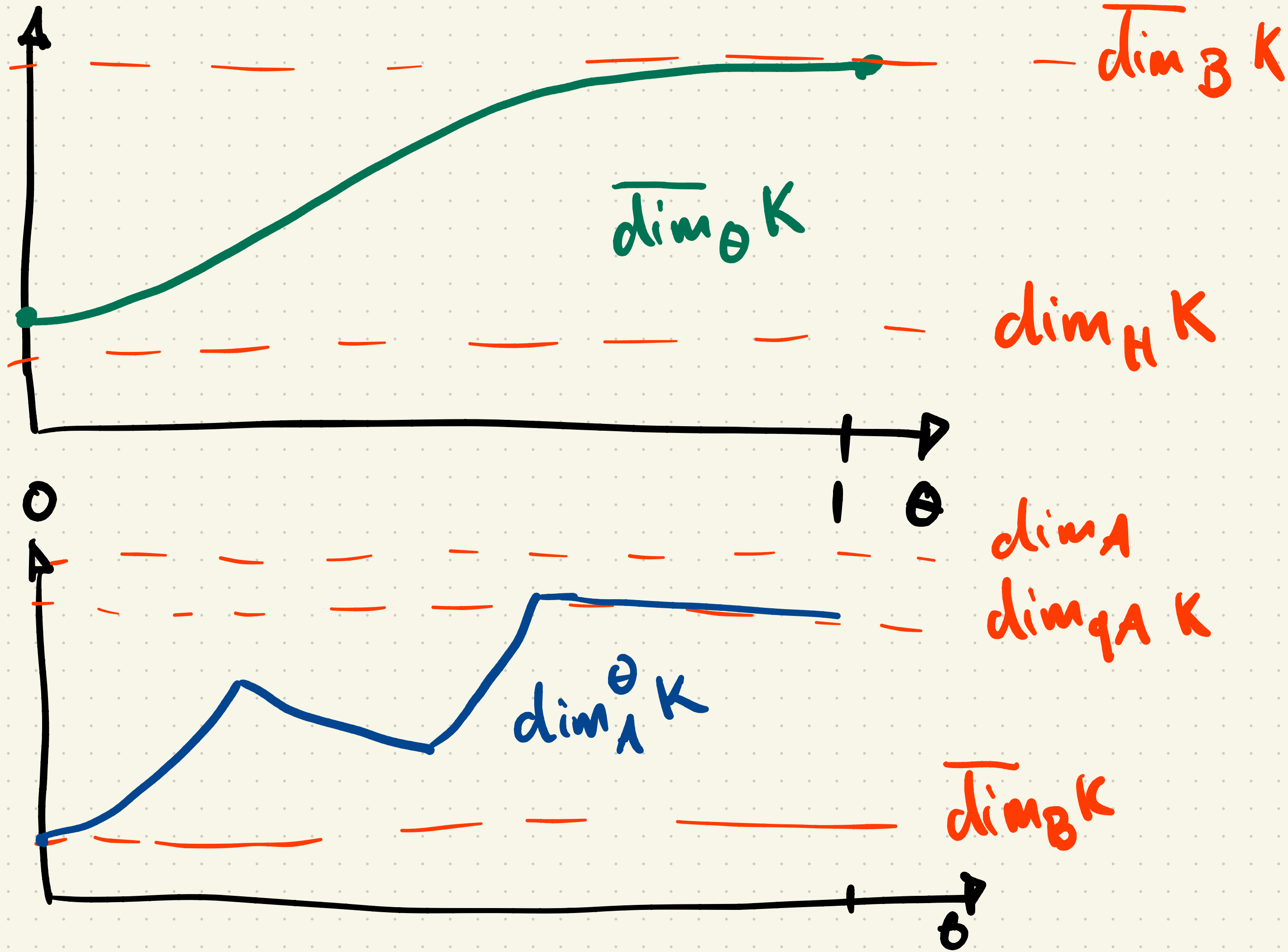
Assouad Spectrum

$$\dim_A K = \inf\left\{ \alpha \geq 0 : \text{for all } 0 < r \leq R \leq 1, \; N_r(B(x,R) \cap K) \lesssim \left(\frac{R}{r}\right)^\alpha \right\}$$

Assouad Spectrum

$$\dim_A^\theta K = \inf\Big\{\alpha \geq 0 : \text{for all } 0 < R \leq 1, \ N_{R^{1/\theta}}(B(x,R)\cap K) \lesssim \left(\frac{R}{R^{1/\theta}}\right)^\alpha\Big\}$$

[for $\theta \in (0,1)$]



$\overline{\dim}_B K$
$\overline{\dim}_\theta K$
$\dim_H K$
0
1
θ
$\dim_A$
$\dim_{qA} K$
$\dim_A^\theta K$
$\overline{\dim}_B K$
θ

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- natural fractal dimensions,
  (bi-Lipschitz invariant, finite stability, etc.)

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- natural fractal dimensions, (bi-Lipschitz invariant, finite stability, etc.)
- Assouad spectra: quasiconformal distortion (Tyson-Garitsis: sharp for polynomial spirals), L^p-L^q distortion bounds (Roos-Seeger)

- Intermediate dimensions: finer information for projections (Burrell + Falconer + Fraser), random images (Burrell)

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- (and more to come!)

Natural Question:

What are possible forms of

$$\theta \longmapsto \overline{\dim}_{\theta} K$$

$$\theta \longmapsto \dim_{A}^{\theta} K$$

for arbitrary Borel $K \subset \mathbb{R}^d$?

Full Characterization for intermediate dimensions

Theorem (Banaji + AR): T.F.A.E

- $\exists K \subset \mathbb{R}^d$ s.t. $\overline{\dim}_\theta K = h(\theta)$
- $0 \leq D^+ h(\theta) \leq \dfrac{h(\theta)(d - h(\theta))}{d\theta}$

$$D^+ h(\theta) = \limsup_{\varepsilon \downarrow 0} \frac{h(\theta+\varepsilon) - h(\theta)}{\varepsilon}$$

Comments:

- if $f:(0,1)\to\mathbb{R}$ is increasing + Lipschitz, $\exists\ a>0,\ b\in\mathbb{R},\ K\subset\mathbb{R}^d$
$$\dim_\theta K = a\cdot f(\theta)+b$$
(very general!)

Consequences:

- if $f:(0,1) \to \mathbb{R}$ is increasing + Lipschitz, $\exists\, a > 0,\ b \in \mathbb{R},\ K \subset \mathbb{R}^d$

$$\dim_\theta K = a \cdot f(\theta) + b$$

(very general!)

- more general form simultaneously characterizing upper + lower, and accounting for Assouad + lower dimensions

Full Characterization for Assouad spectra

Theorem (AR) T.F.A.E.

- $\exists K \subset \mathbb{R}^d$ s.t. $\dim_A^\theta = \varphi(\theta)$
- For all $0 < \lambda < \theta < \lambda^{1/2} < 1$,

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta-\lambda)\varphi\left(\frac{\lambda}{\theta}\right)$$

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- $\leq$ holds $\Longleftrightarrow$ $D^+\varphi(\theta) \leq \dfrac{\varphi(\theta)}{1-\theta}$
- if φ increasing, $\leq$ always holds

  [N.B. $\dim_\lambda^\theta K$ not monotonic, in general]

$\Rightarrow$ $\leq$ is an "osscilation" bound

Monotonicity?

Question (Fraser): does Assouad spectrum need to be monotonic in neighbourhood of 1?

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Theorem (AR) NO. In fact, Assouad spectra which are non-monotonic on every open set are uniformly dense in all possible Assouad spectra.

Bounding Intermediate Dimensions

- Bound $\dim_{\theta+\varepsilon} K$ in terms of $\dim_\theta K$.

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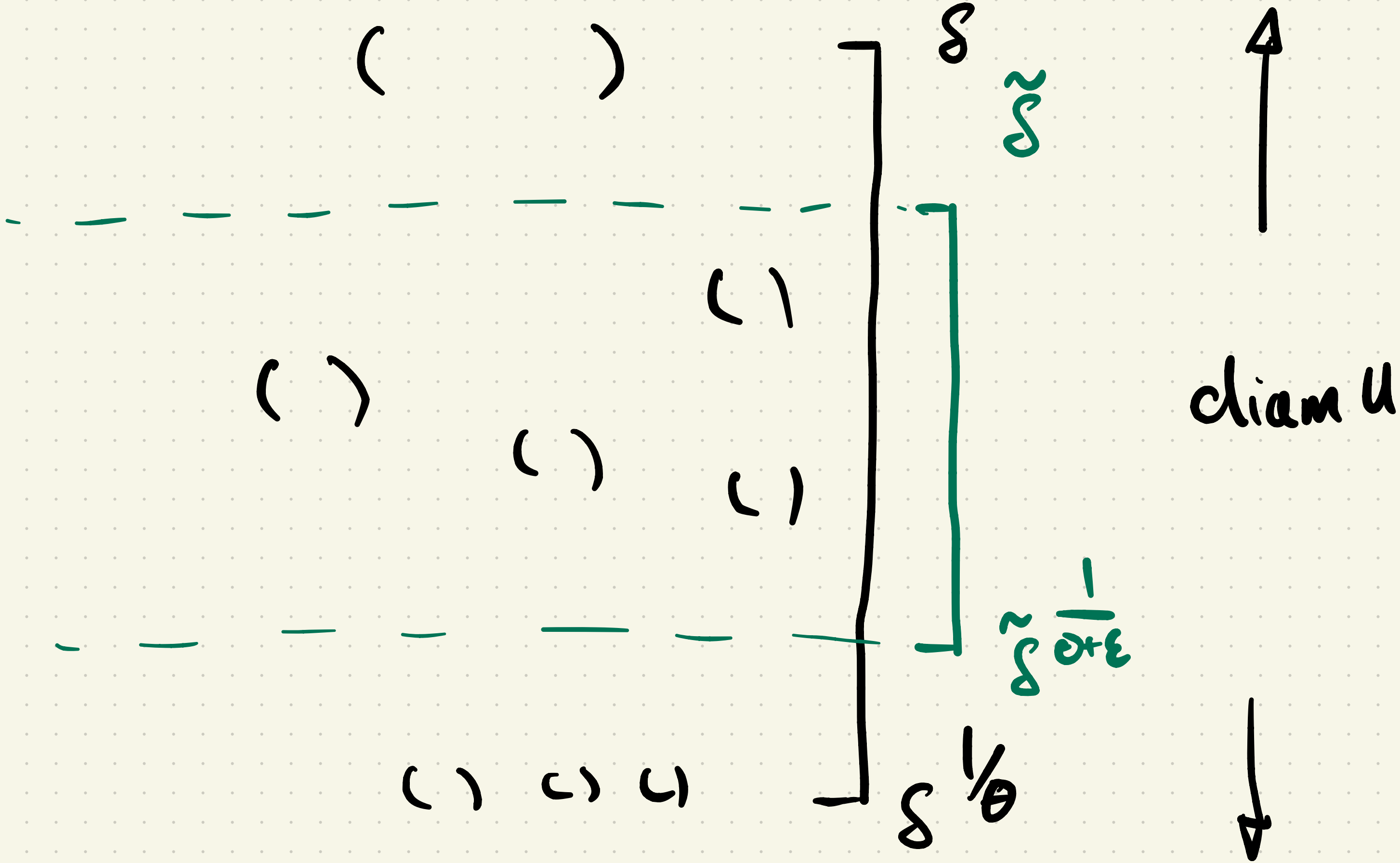
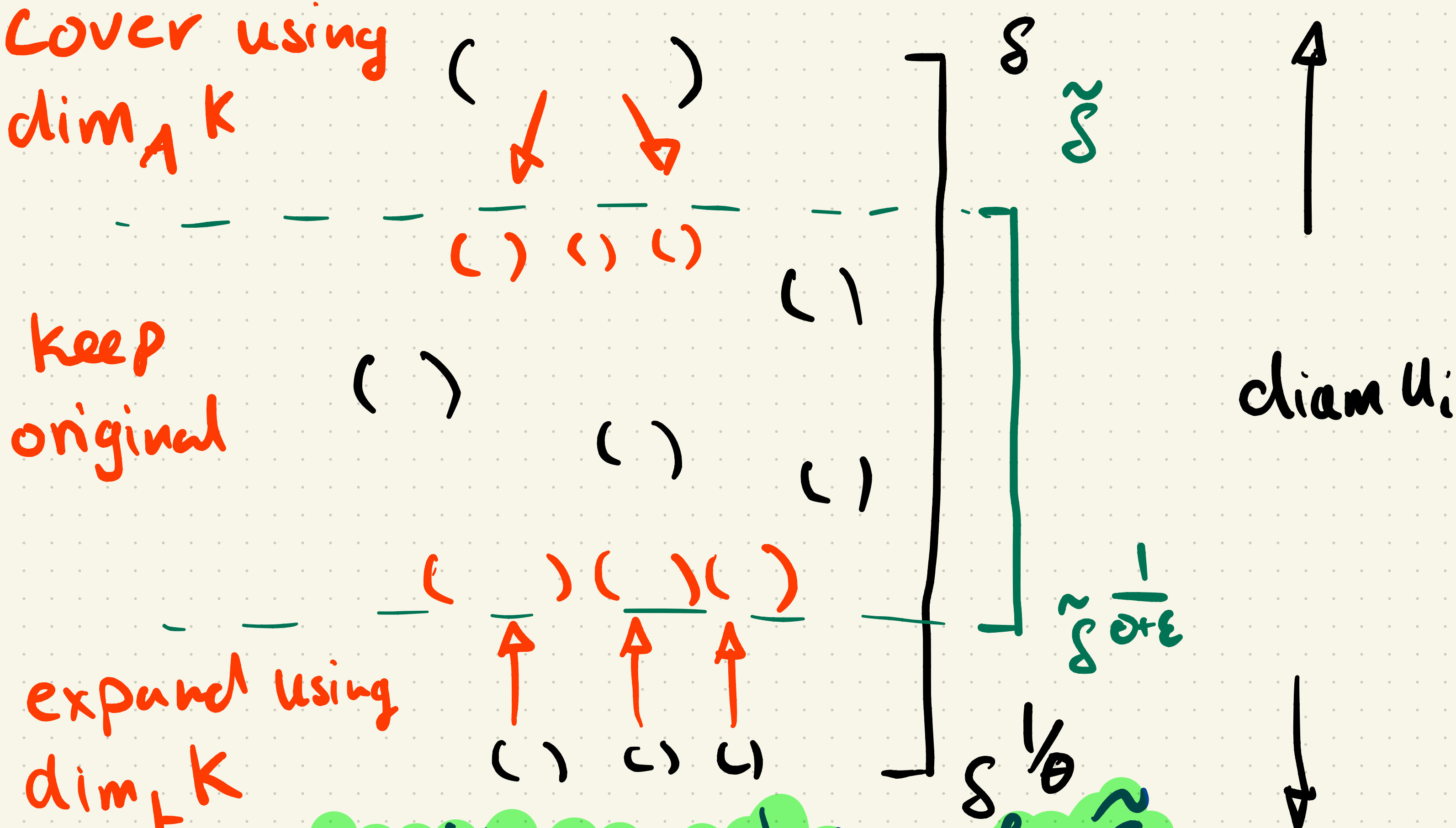
  - $\hookrightarrow$ convert cover

$\delta^{1/\theta} \leq \operatorname{diam} U_i \leq \delta$ to $\delta^{\frac{1}{\theta+\varepsilon}} \leq \operatorname{diam} U_i \leq \delta$



δ
$\tilde{\delta}$
$\tilde{\delta}^{\frac{1}{\theta+\varepsilon}}$
$\delta^{1/\theta}$
diam U_i



Optimize choice of $\tilde{\delta}$.

Bounding Assouad spectra

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$\leqslant$ $B(x, \delta^{1/\theta}) \subseteq B(x, \delta^{1/\lambda})$ so

$$\sup_x N_\delta(B(x, \delta^{1/\theta}) \cap K) \leqslant \sup_x N_\delta(B(x, \delta^{1/\lambda}) \cap K)$$

+ algebra

Bounding Assouad spectra

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta-\lambda)\varphi\left(\frac{1}{\theta}\right)$$

$\leq$ cover $B(x, \delta^{\lambda})$ w/ balls $B(y, \delta^{\theta})$

$$\sup_{x\in K} N_{\delta}(B(x,\delta^{\lambda}) \cap K) \leq \sup_{x\in K} N_{\delta^{\theta}}(B(x,\delta^{\lambda}) \cap K) \cdot \sup_{x\in K} N_{\delta}(B(x,\delta^{\theta}) \cap K)$$

+ algebra

Moran Constructions

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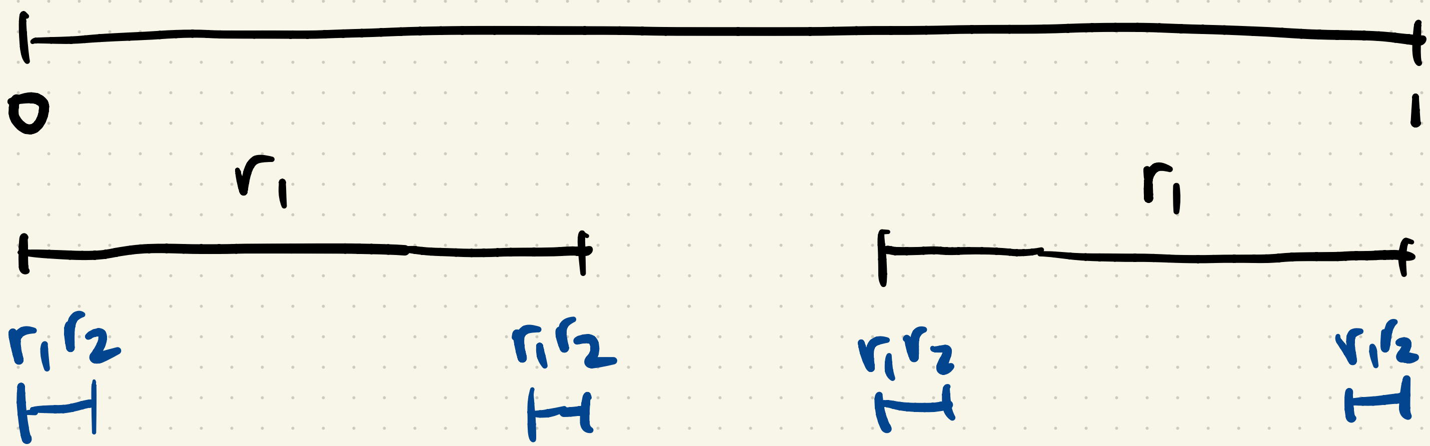

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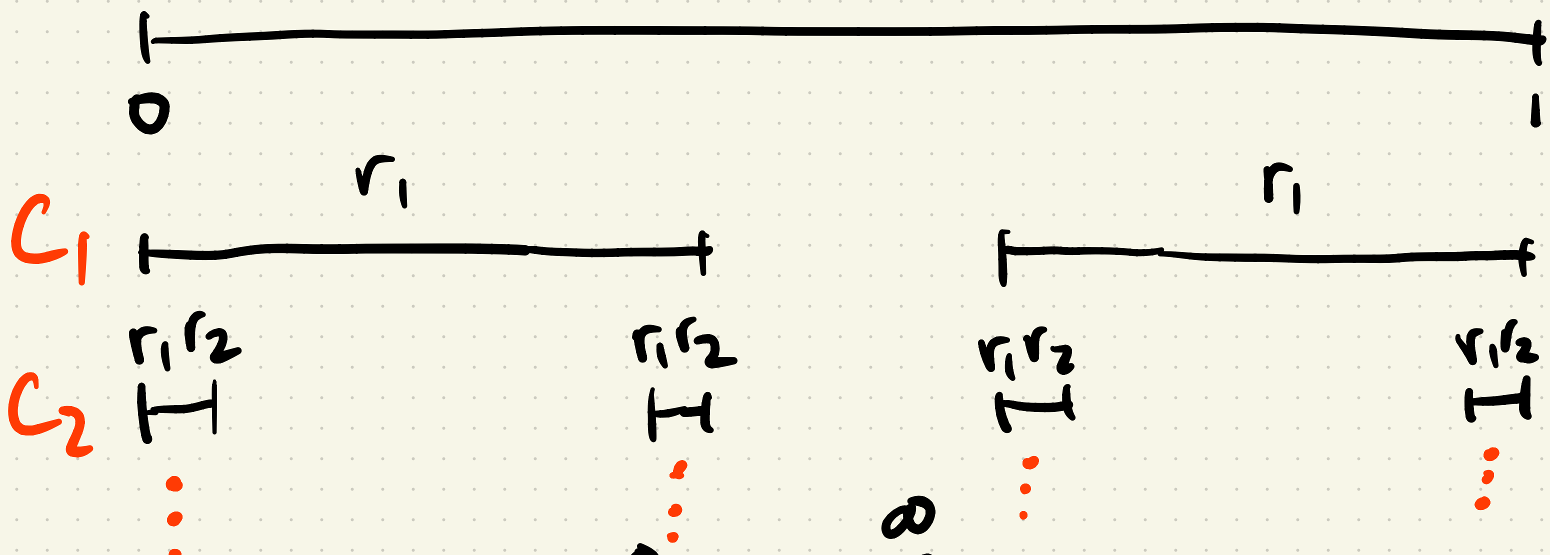

$$C(r_n)_{n=1}^{\infty} = \bigcap_{n=1}^{\infty} C_n$$

Lemma (Banaji + AR) Let $g:(0,\infty) \to [0,d]$.

s.t. $D^+ g(x) \in [-g(x), d - g(x)]$.

Then there exists homogeneous Moran set so that

$$\left| S(\exp(-\exp(x))) - g(x) \right| \leq d \log 2 \exp(-x)$$

where $s(\delta) = \dfrac{k(\delta)\, d \log 2}{-\log \delta}$ w/ $r_1 \cdots r_{k(\delta)} \leq \delta < r_1 \cdots r_{k(\delta)-1}$

$$\left| s(\exp(-\exp(x))) - g(x) \right| \leq d \log 2 \exp(-x)$$

- $s(\delta)$ = "box dimension at scale δ"

(in fact $\underline{\dim}_B C = \liminf_{\delta \to 0} s(\delta)$

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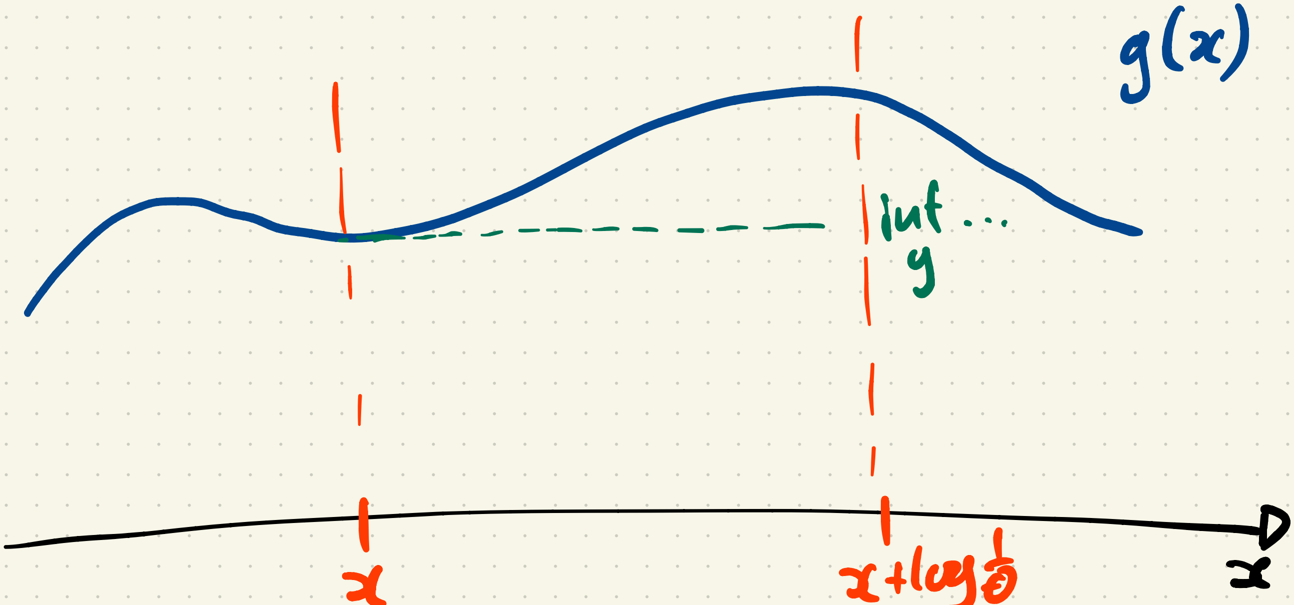
- double log rescaling: $\delta^{1/\theta} \sim x + \log \frac{1}{\theta}$.

Formulas for $\dim_\theta C$, $\dim_A^\theta C$

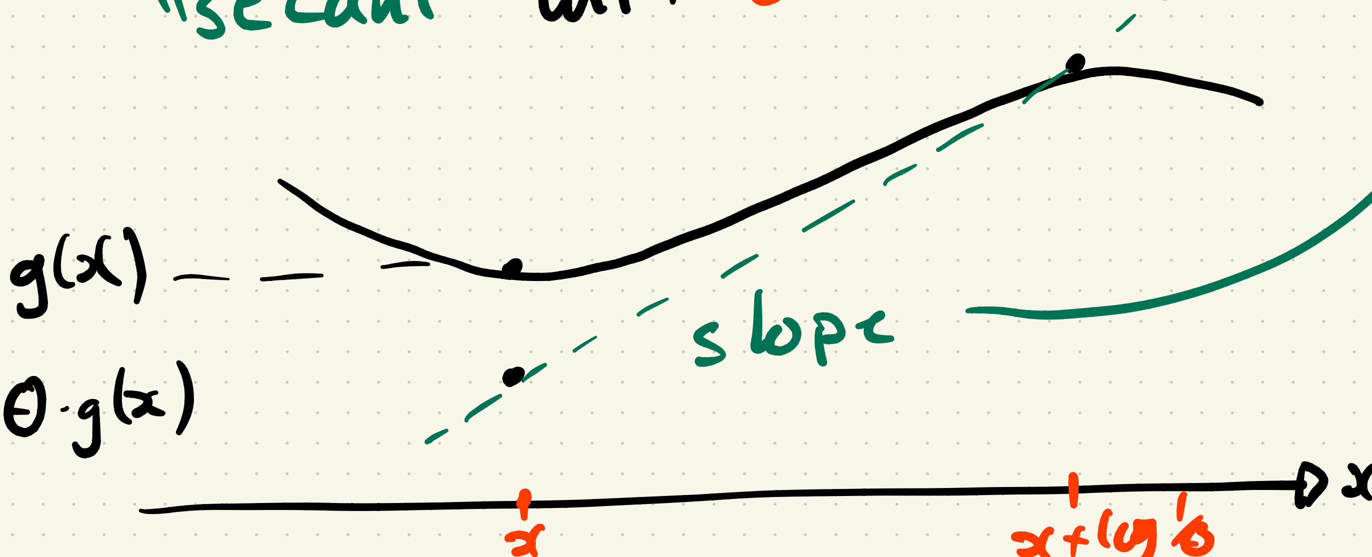
- $\dim_\theta K = \limsup_{x \to \infty} \left[\inf_{y \in [x, x+\log\frac{1}{\theta}]} g(x) \right]$



Formulas for $\dim_\theta C$, $\dim_A^\theta C$

- $\dim_A^\theta K = \limsup_{x\to\infty}\left[\dfrac{g(x+\log\frac{1}{\theta}) - \theta g(x)}{1-\theta}\right]$

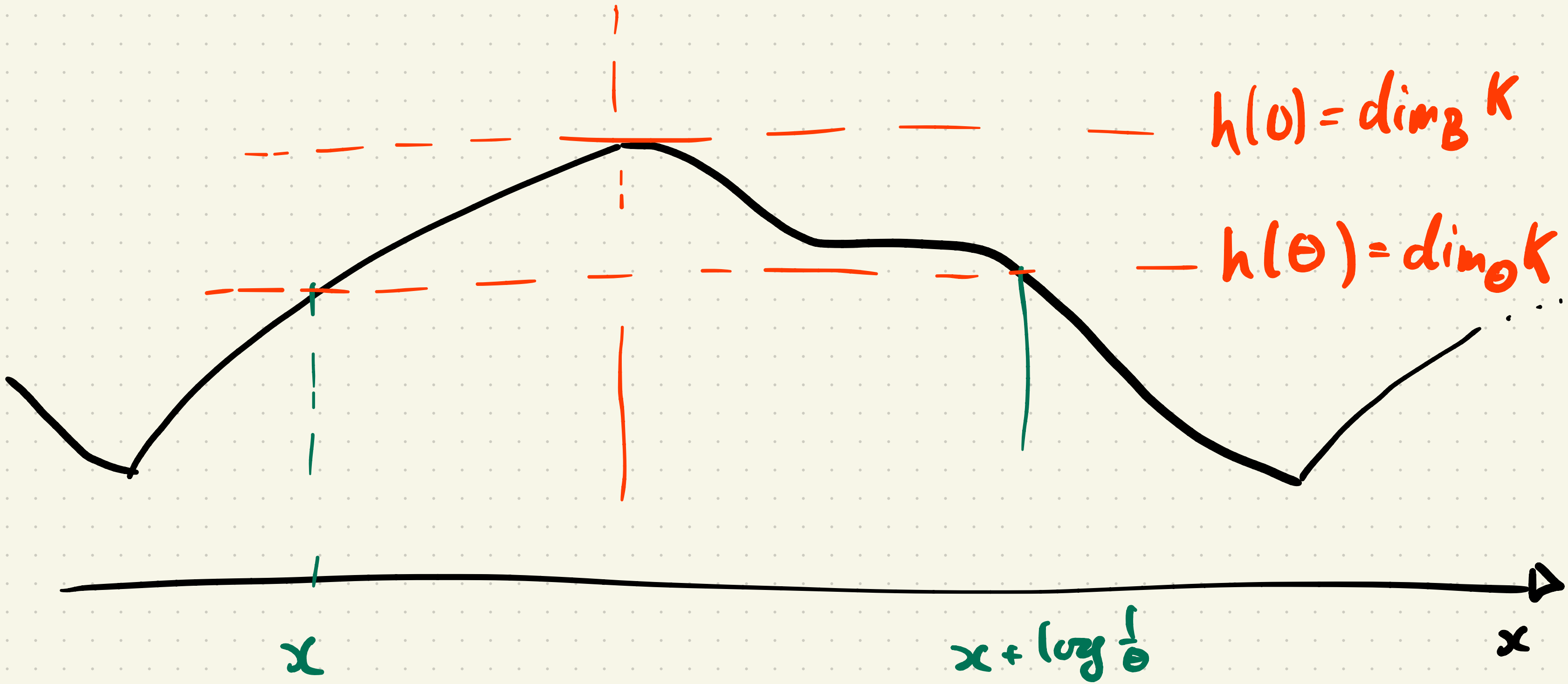
"secant" with "scale-invariance"



To prove sharpness of bounds:

$$\boxed{\text{Choose } g(x) \text{ carefully.}}$$

"Bump Construction" for Inf. Dims,



Construction for Assouad spectrum.