Geometric and Combinatorial Properties of Self-similar Measures

Alex Rutar // University of St Andrews

FARF 2022

Multifractal Analysis

μ: probability measure (compactly supported, Borel)

Goal: Understand "scaling" / "size" of μ

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$x \in \operatorname{supp} \mu$, local dimension:

$$\dim_{loc}(\mu, x) = \lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

(when the limit exists)

$$\lim_{r \to 0} \frac{\log \mathcal{N}(B(x, r))}{\log r}$$

- exponential growth rate of measure vs. size of ball

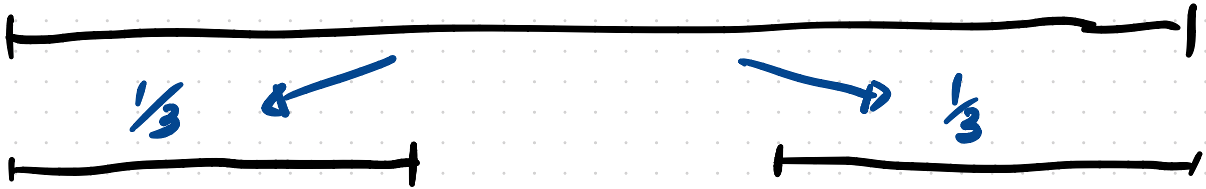
$$\lim_{r \to 0} \frac{\log \mu(B(x,r))}{\log r}$$

- exponential growth rate of measure vs. size of ball
- x is an atom: $\mu(B(x,r)) \geq C > 0$

  $\Rightarrow \dim_{loc}(\mu, x) = 0$

$$\lim_{r\to 0} \frac{\log \nu(B(x,r))}{\log r}$$

- exponential growth rate of measure vs. size of ball
- x is an atom: $\nu(B(x,r)) \geq C > 0$
  $\Rightarrow \dim_{loc}(\nu,x) = 0$
- ν is Lebesgue $(\mathbb{R}^d)$: $\nu(B(x,r)) \approx r^d$
  $\Rightarrow \dim_{loc}(\nu,x) = d$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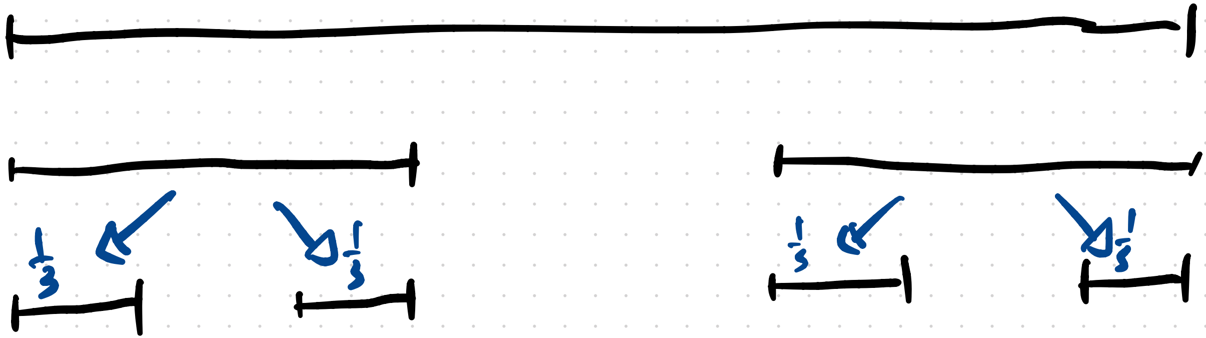
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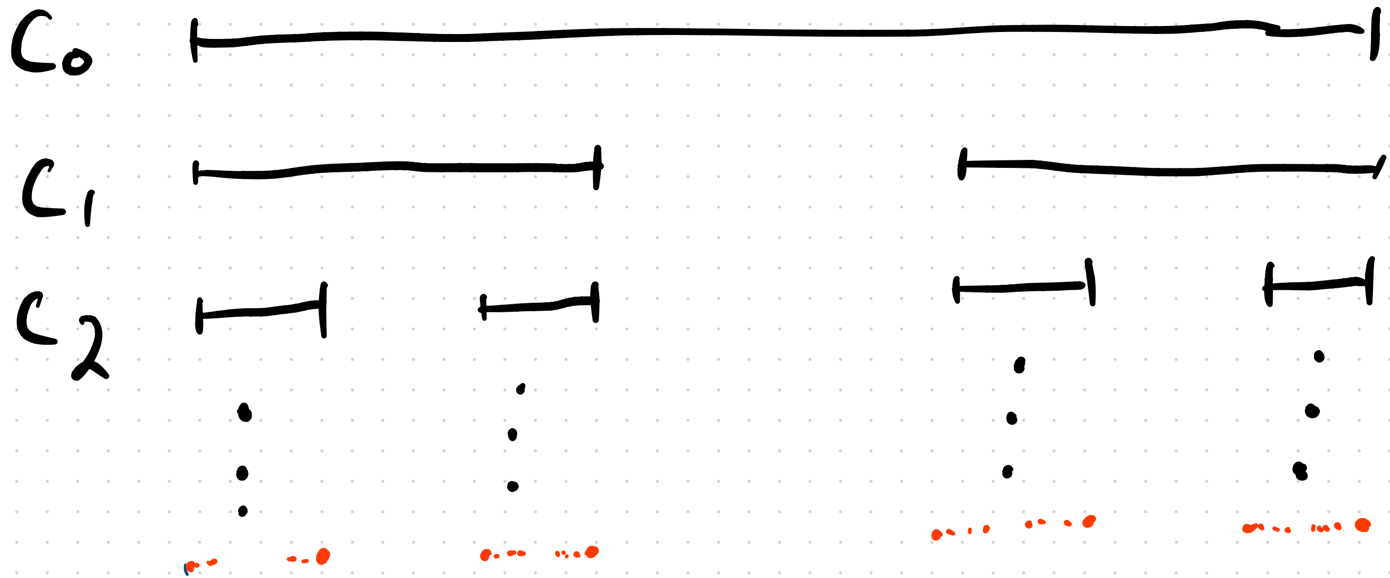
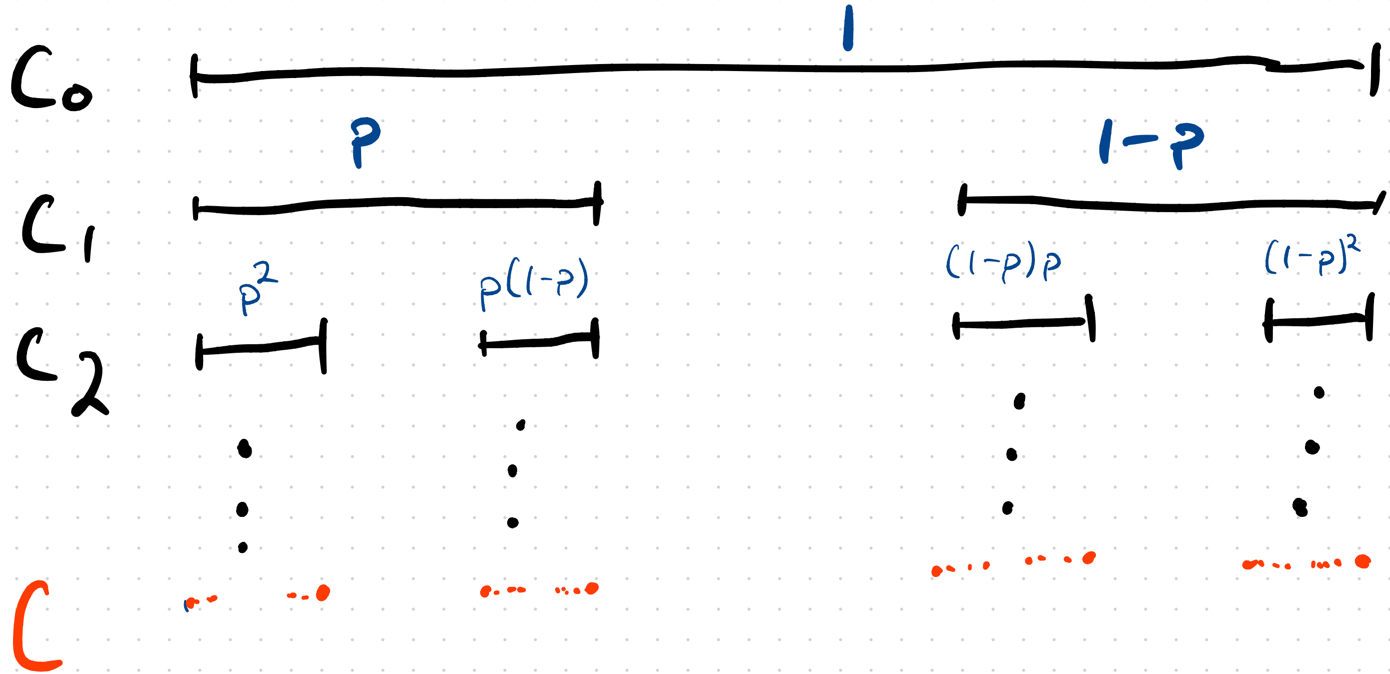


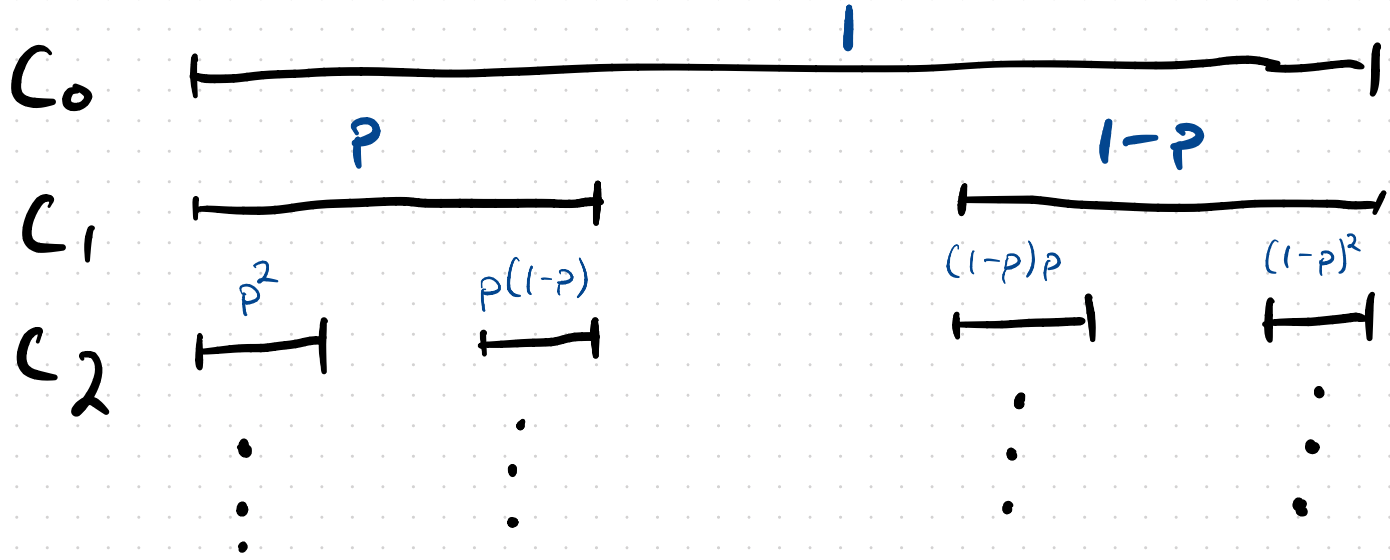
$$C = \bigcap_{n=1}^{\infty} C_n$$

Cantor set



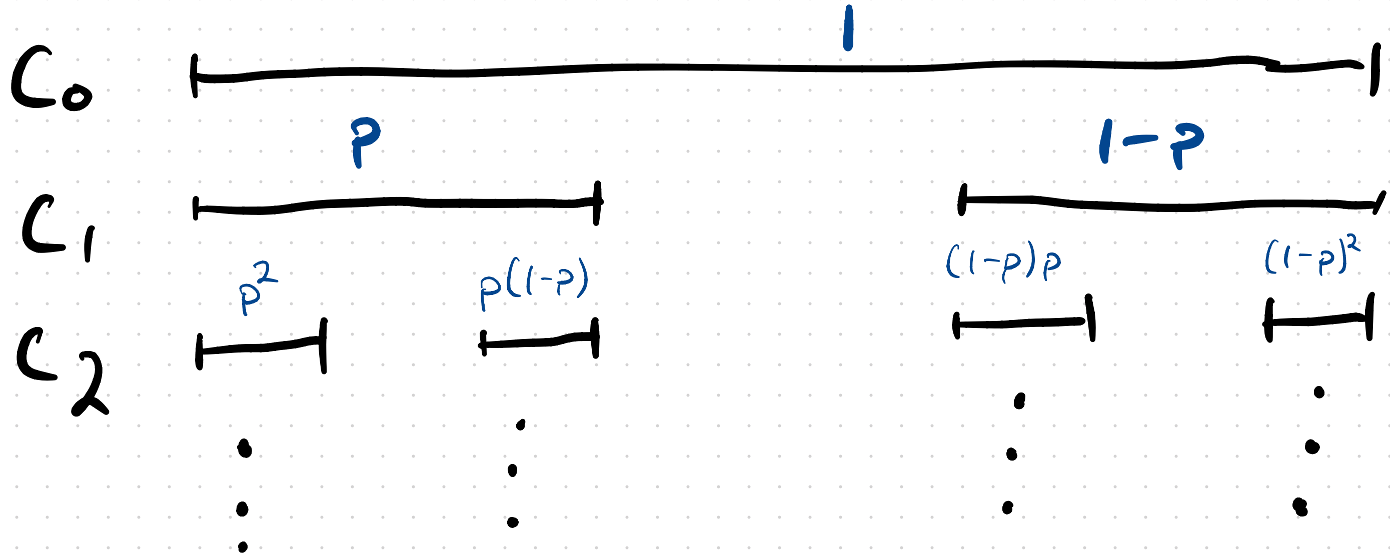
$p \in (0,1) \rightsquigarrow$ Cantor measure ν_p

$\operatorname{supp} \nu_p = C$



$p = \frac{1}{2}$: $B(x, 3^{-n}) \cap K =$ level n interval

$\Rightarrow \mu(B(x, 3^{-n})) = 2^{-n}$

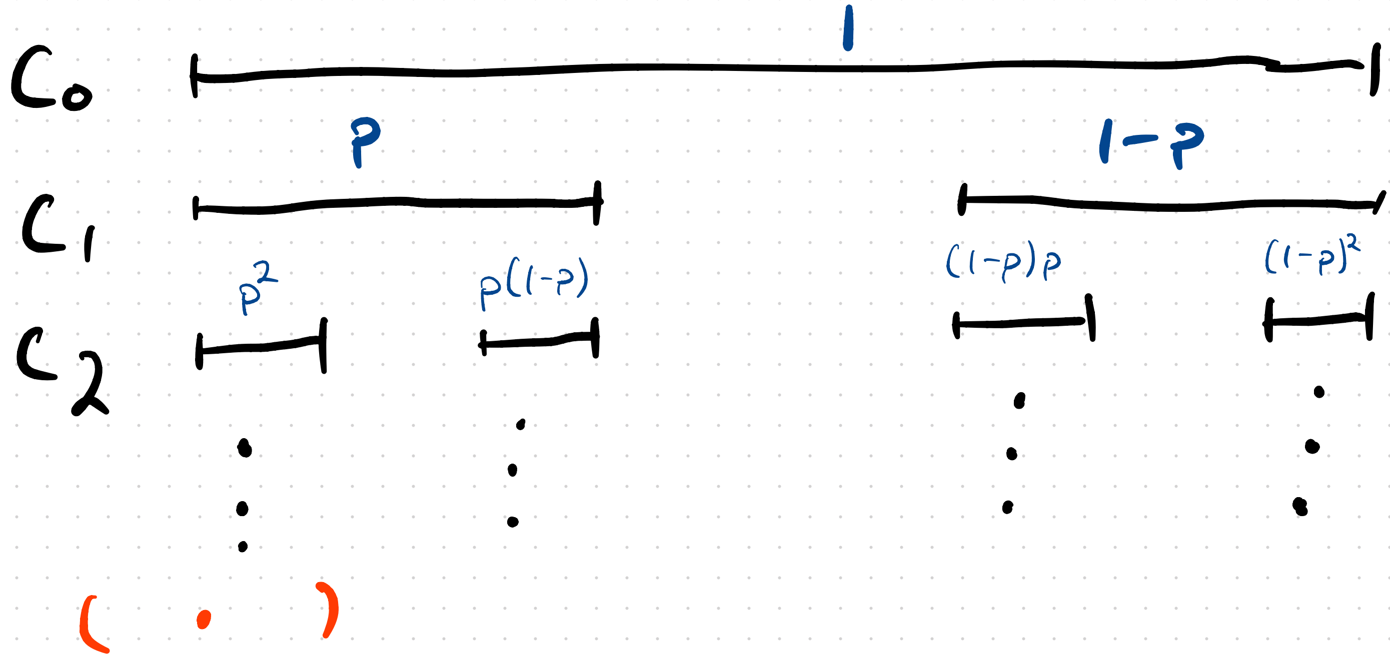


$p = \frac{1}{2}$: $B(x, 3^{-n}) \cap K =$ level n interval

for all $x \in C$

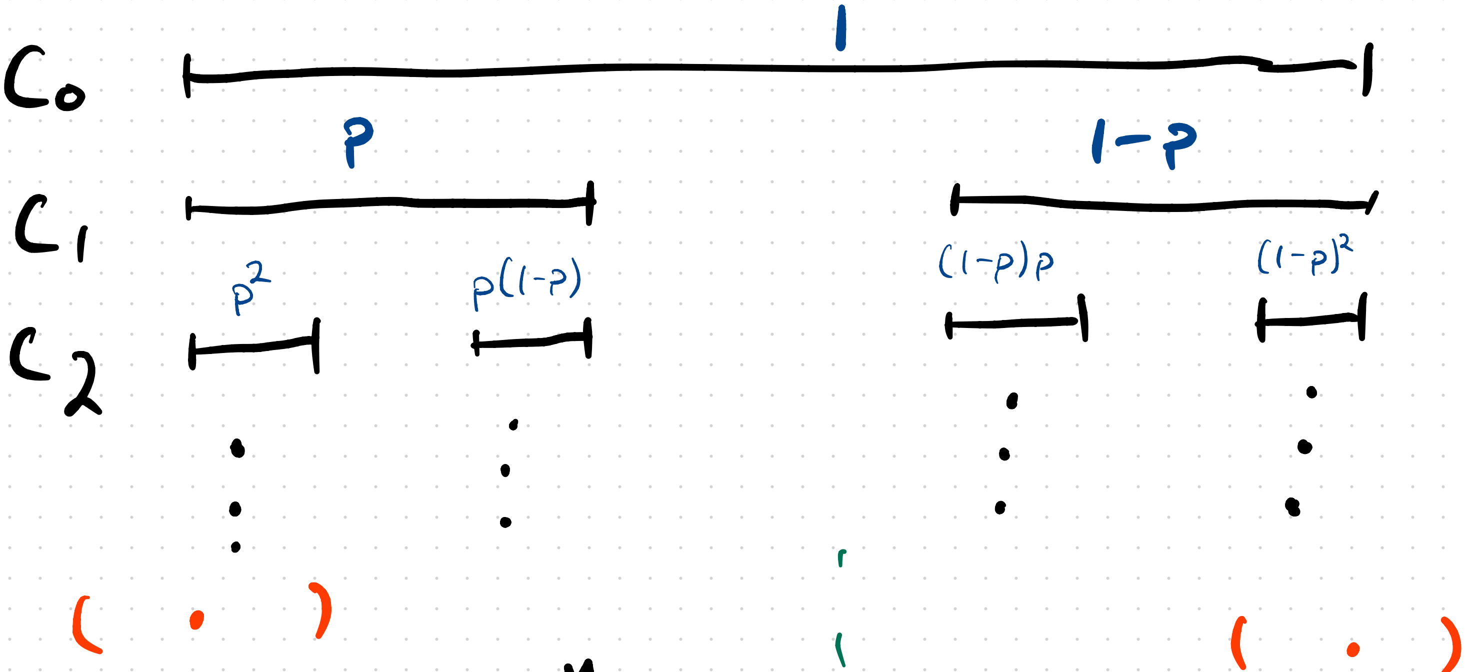
$\Rightarrow \nu(B(x, 3^{-n})) = 2^{-n}$

$$\dim_{loc}(\nu, x) = \lim_{n} \frac{\log \nu(B(x, 3^{-n}))}{\log 3^{-n}} = \frac{\log 2}{\log 3}$$



$$\mu(B(0,3^{-n})) = p^n$$

$$\Rightarrow \dim_{loc}(\mu, 0) = \frac{\log p}{\log \frac{1}{3}}$$

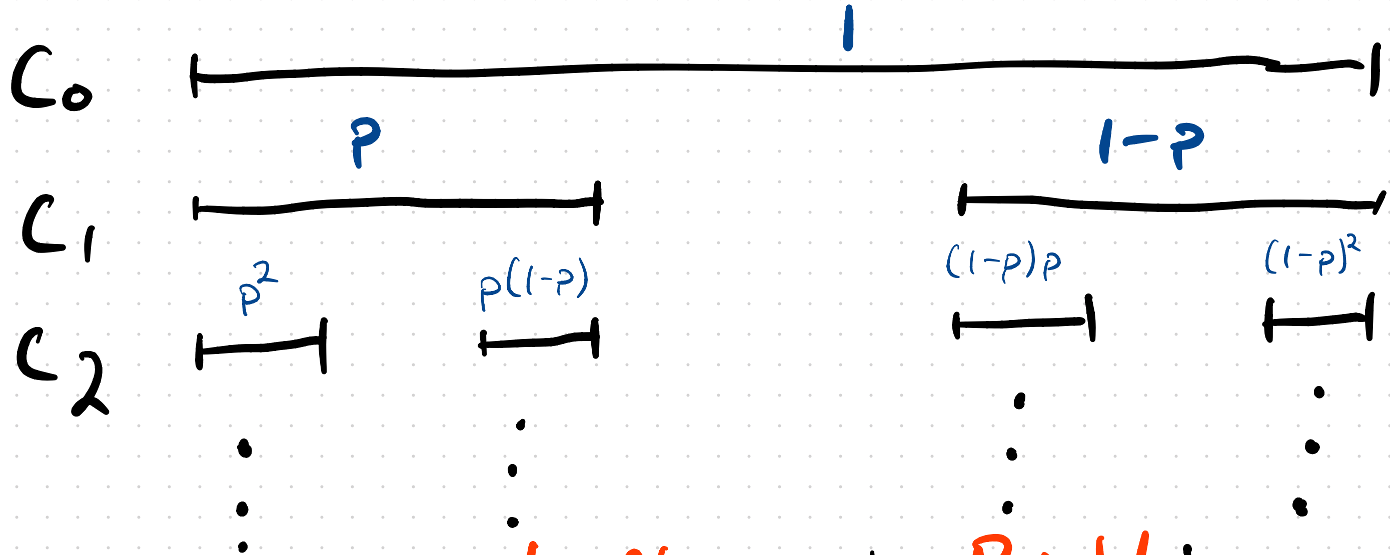


$$\nu(B(0,3^{-n})) = p^n$$

$$\Rightarrow \dim_{loc}(\nu, 0) = \frac{\log p}{\log \frac{1}{3}}$$

$$\nu(B(1,3^{-n})) = (1-p)^n$$

$$\Rightarrow \dim_{loc}(\nu, 1) = \frac{\log (1-p)}{\log \frac{1}{3}}$$



mixing Left and Right:

$$\forall \alpha \in \left[\frac{\log p}{\log \frac{1}{3}}, \frac{\log (1-p)}{\log \frac{1}{3}} \right]$$

$$\exists x \in \operatorname{supp} \nu : \dim_{loc}(\nu, x) = \alpha$$

Multifractal Analysis: Understand

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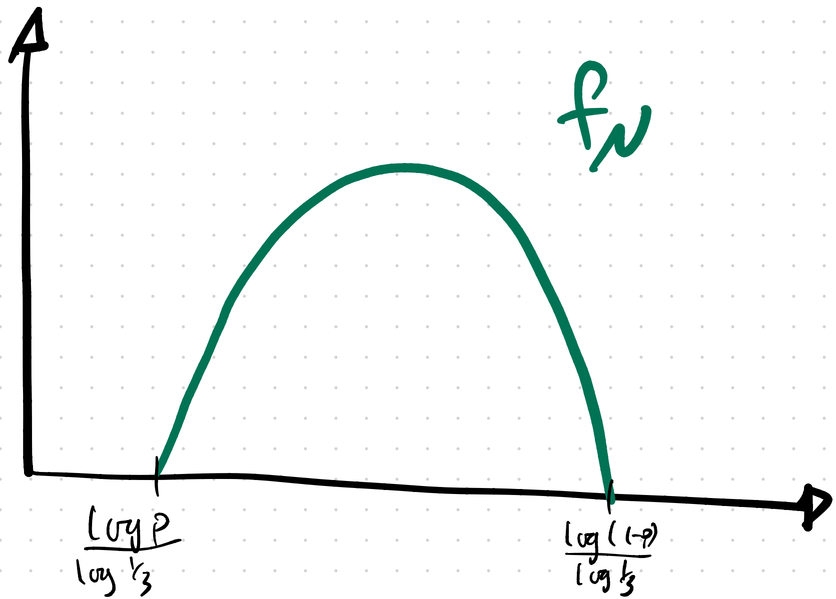
- $K_\nu(\alpha) = \{x \in \operatorname{supp} \nu : \dim_{loc}(\nu, x) = \alpha\}$

- $f_\nu(\alpha) = \dim_H K_\nu(\alpha)$

"multifractal spectrum"

Cantor measure ν_p:

- $K_{\nu_p}(\alpha) = \left[\frac{\log p}{\log \frac{1}{3}} , \frac{\log (1-p)}{\log \frac{1}{3}} \right]$



Self-similar measures

- $\{S_i\}_{i\in\mathcal{I}}$: $S_i(x) = r_i x + d_i : \mathbb{R} \to \mathbb{R}$

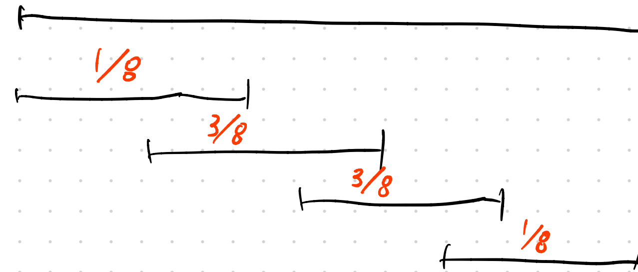
  $0 < |r_i| < 1$ (Lipschitz contractions)

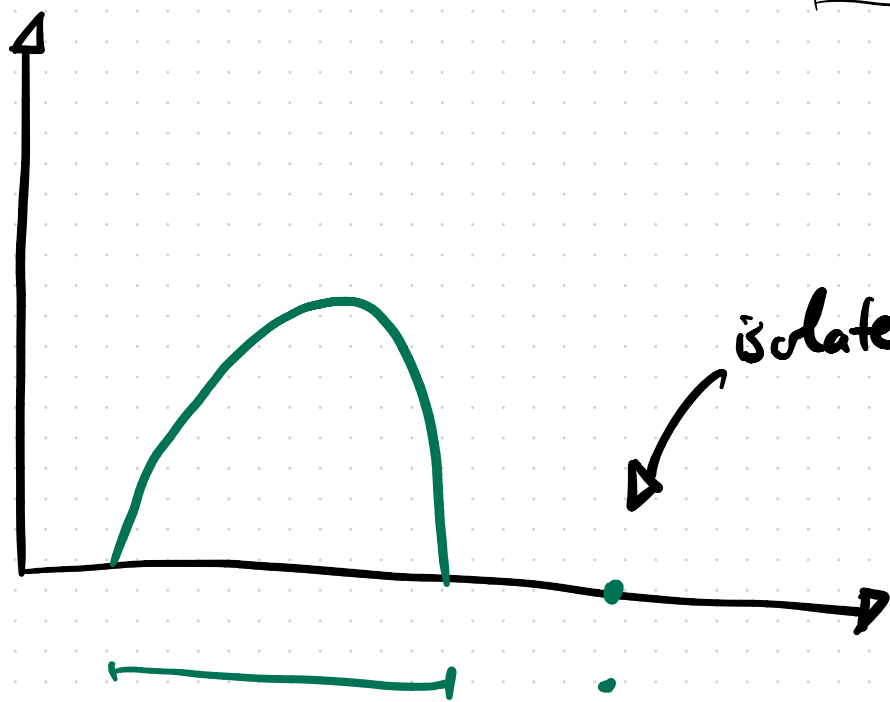
- $(p_i)_{i\in\mathcal{I}}$: $p_i > 0$, $\sum_i p_i = 1$

$\exists$ unique measure ν with

$$\nu(E) = \sum_{i\in\mathcal{I}} p_i \, \nu(S_i^{-1}(E))$$

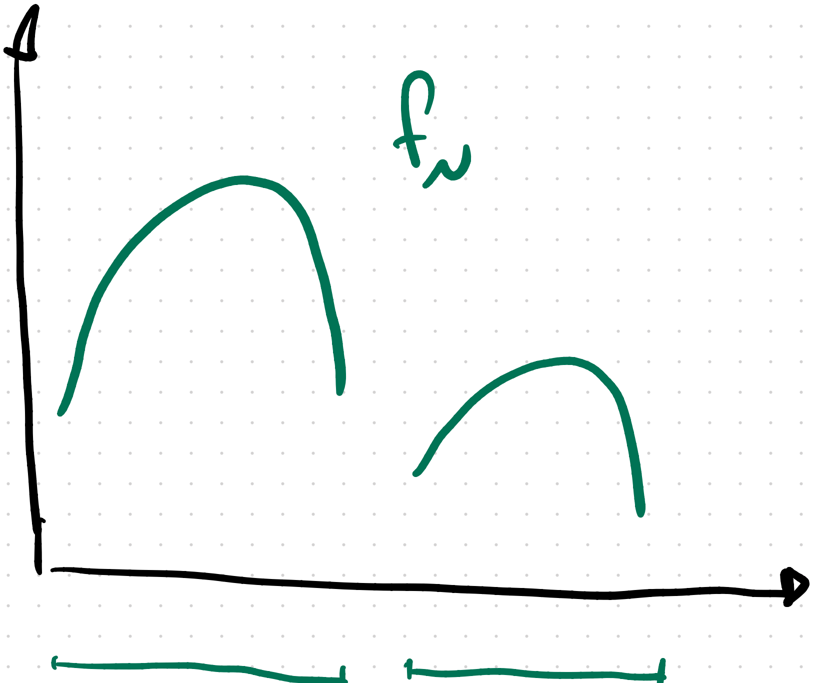
Hu-Lau, 2002

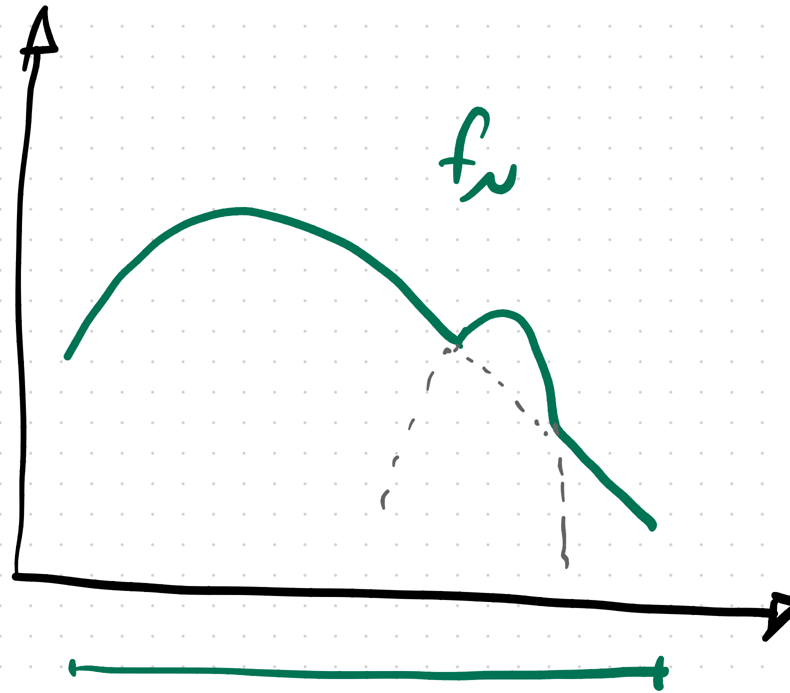




Testud, 2002

"digit" IFS $x \mapsto \pm \frac{x}{\ell} + \frac{j}{\ell}$





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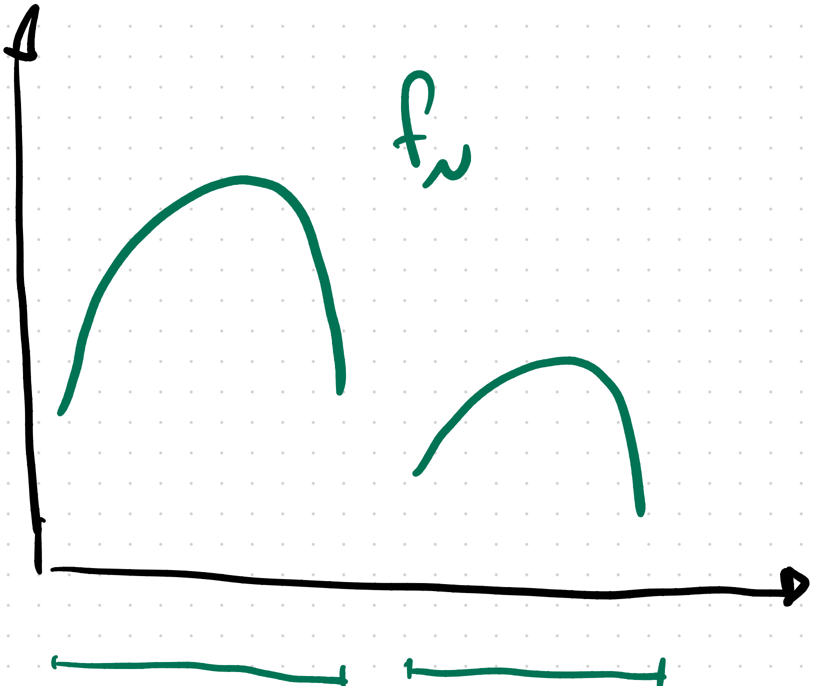

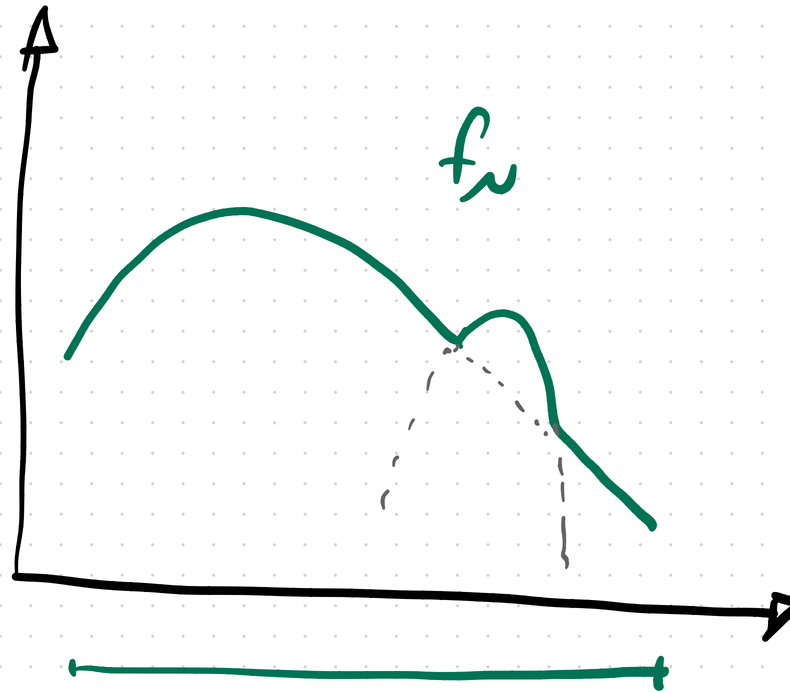

(and many other examples!)

General explanation for this phenomenon?

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- Hu-Lau, Tested examples : satisfy finite type condition

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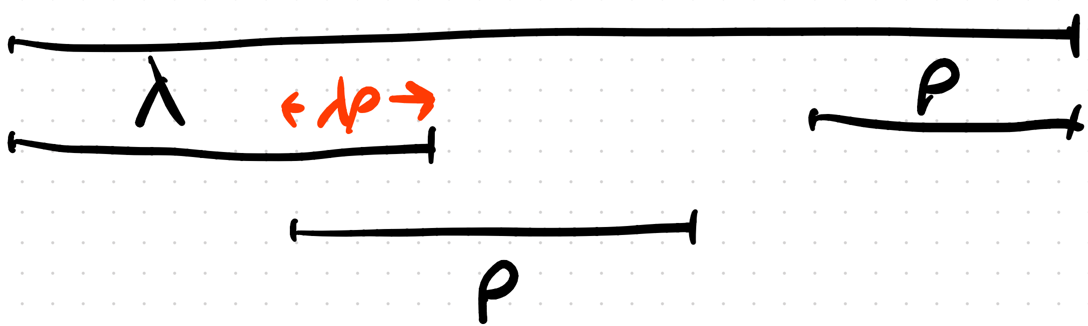
- 0-1-3 measure, and inhomogeneous generalization:



- net interval construction (Feng 2003, AR 2022)

$\Rightarrow$ construct symbolic space with (almost unique) coding, and measure given by products of matrices

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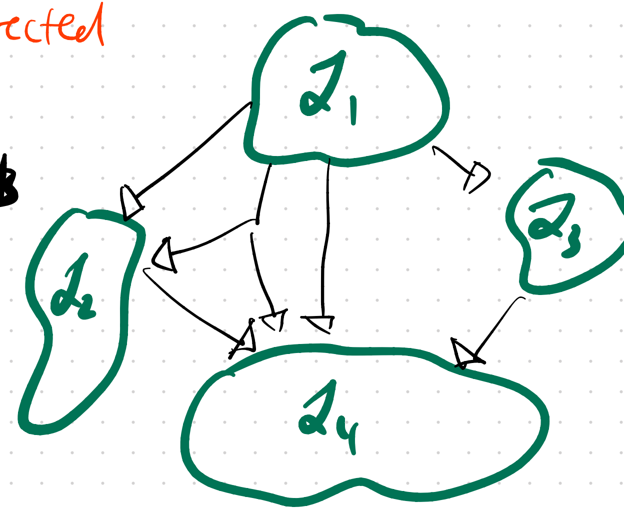
- construct graph G, directed

  maximal connected components

  $\mathcal{L}_1, \ldots, \mathcal{L}_n$



Theorem (AR) Let $\{S_i\}_{i \in \mathcal{I}}$ be an IFS in $\mathbb{R}$, satisfying finite type condition, with graph G, components $\mathcal{L}_1, \ldots, \mathcal{L}_n$.

Suppose components are irreducible. Then

$$f_\nu = \max\{f_1, \ldots, f_n\}$$

where each f_i is concave and satisfies a multifractal formalism.

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Suppose components are irreducible. only needed for $n \geq 2$ Then

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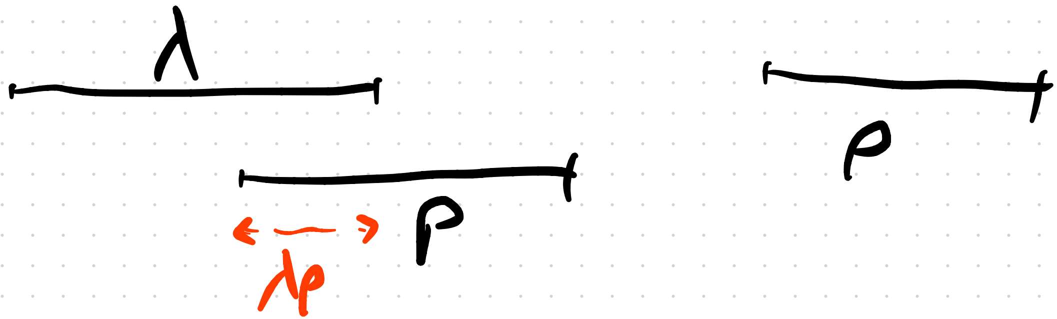
(3) FNC $\subseteq$ WSC (and conjecturally equal: Feng 2016, AR+Hare^2 2021)

Corollary: IFS



satisfies multifractal formalism for all $q \in \mathbb{R}$, any probabilities.

Questions: Let ν be self-similar.

(1) Is f_ν always a finite maximum of concave functions?

(2) If f_ν is concave, does ν satisfy multifractal formalism?