Self-similar measures with non-concave spectra and multifractal analysis

Alex Rutar / JETS April 16, 2021

1. Multifractal spectrum

μ Borel probability measure on $\mathbb{R}^{(d)}$, compact support.

> Question: What does the "density" of μ "look like" w.r.t. Lebesgue measure $(= m)$?

If $\mu \ll m$, $\mu = \int f \, dm$. f "encodes density" of μ.

1. What can we say about the function f?
2. What if $\mu \not\ll m$?

Definition: The local dimension of ν at $x \in \operatorname{supp} \nu$

$$\dim_{loc}(\nu, x) = \lim_{r \to 0} \frac{\log \nu(B(x,r))}{\log r}$$

"Exponential decay rate of $\nu(B(x,r))$ w.r.t. $m(B(x,r)) \approx r$"

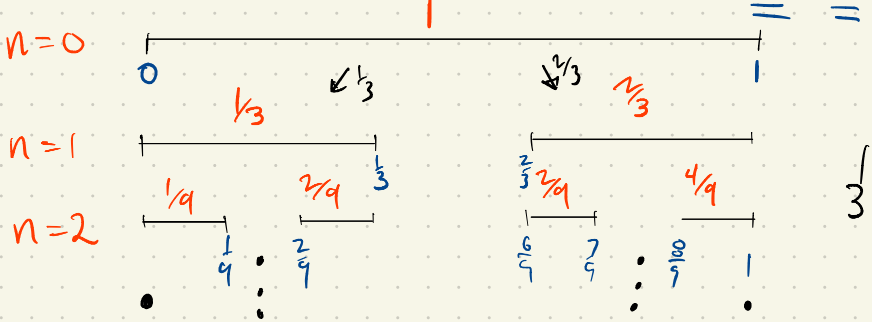
Ex. If ν = Lebesgue on $[0,1]$, $x \in [0,1]$

$$r \leq \nu(B(x,r)) \leq 2r$$

$$\Longrightarrow \dim_{loc}(\nu, x) = \lim_{r \to 0} \frac{\overbrace{\log r}^{\approx \log 2r}}{\log r} = 1.$$

Same true if $\nu \ll m$ with $\underbrace{\text{"nice"}}_{\text{e.g. piecewise Lipschitz}}$ density.

Ex. ν = Cantor measure, probabilities $\frac{1}{3}, \frac{2}{3}$:



Divide mass $\frac{1}{3}$ on left, $\frac{2}{3}$ on right

$\operatorname{supp} \nu$ = Cantor set.

$$\nu(B(0, 3^{-n})) = \left(\tfrac{1}{3}\right)^n$$

$$\Longrightarrow \dim_{loc}(\nu, 0) = \lim_{n\to\infty} \frac{\log 3^{-n}}{\log 3^{-n}} = 1$$

$$\nu(B(1, 3^{-n})) = \left(\tfrac{2}{3}\right)^n$$

$$\Longrightarrow \dim_{loc}(\nu, 1) = \lim_{n\to\infty} \frac{\log \left(\frac{3}{2}\right)^{-n}}{\log 3^{-n}} = \frac{\log 3/2}{\log 3} < 1$$

By "mixing", get $\dim_{loc}(\nu, x) = \alpha$ for any

$$\alpha \in \left[-\frac{\log 2/3}{\log 3}, -\frac{\log 1/3}{\log 3} \right] =: D(\nu)$$

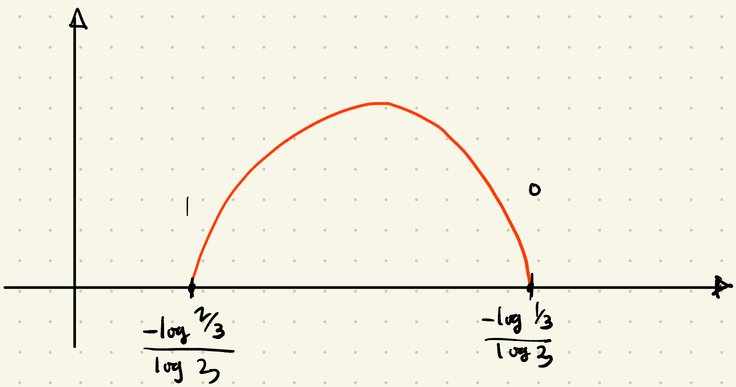
Observation: $\{x : \dim_{loc}(\nu, x) = \alpha\}$ is dense in $\operatorname{supp} \nu$.

Also: some of these sets are "larger" than others.

Definition: the multifractal spectrum of ν

$$f_\nu(\alpha) = \dim_H \{x : \dim_{loc}(\nu, x) = \alpha\}$$

What does f_ν look like? Cantor ex:



- f_ν concave, differentiable, etc.

2. Multifractal formalism

Multifractal spectrum is very "local". Alternative:

Definition: the L^q spectrum of ν is $q \in \mathbb{R}$

$$\tau_\nu(q) = \liminf_{r \to 0} \frac{\log \sup \sum_i \nu(B(x_i,r))^{q}}{\log r}$$

$q=0$ $\sum_i \nu(B(x_i,r))^0$ = size of largest centred packing

w/ supremum over centred packings $\{B(x_i,r)\}_i$ of $\operatorname{supp} \nu$. $x_i \in \operatorname{supp}\nu$

E.g. ν = Lebesgue on $[0,1]$. If $r>0$, fit $\approx \frac{1}{r}$ balls $B(x_i,r)$, each w/ mass

$\nu(B(x_i,r))^q \approx r^q$ $\quad r^{-1} \cdot r^q$

$$\Rightarrow \sup \sum_i \nu(B(x_i,r))^q \approx r^{q-1}$$

$$\Rightarrow \tau_\nu(q) = \liminf_{r\to 0} \frac{\log r^{q-1}}{\log r} = q-1.$$

General Facts

- $\tau_\nu(q)$ is concave, increasing (Hölder's inequality)
- $\dim_B \operatorname{supp}\nu = -\tau_\nu(0)$
- If ν is "nice" (e.g. exact dimensional)

  $\dim_H \nu = \tau_\nu'(1)$. (Heurteaux '98)

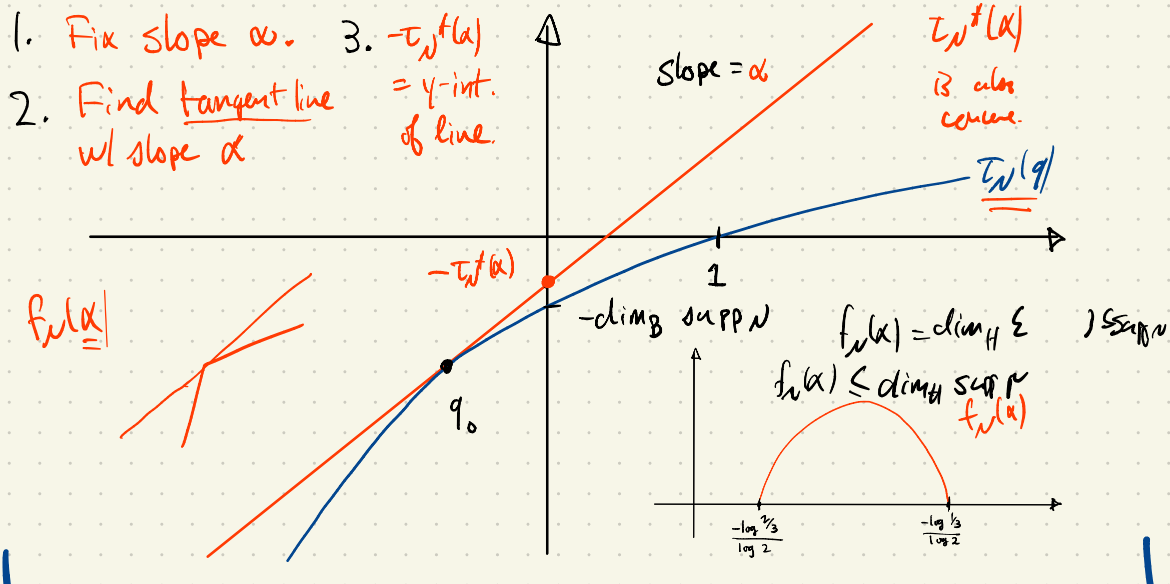
Introduced in (Hentschel et. al., 84')

Definition: ν satisfies **multifractal formalism** if

$$f_\nu(\alpha) = \tau_\nu^*(\alpha) = \inf_{q \in \mathbb{R}} \{ q\alpha - \tau(q) \}$$

In particular: $f_\nu(\alpha)$ is concave function.

Ex. Cantor measure w.r.t. probabilities $\frac{1}{3}, \frac{2}{3}$.



Question: how "generic" is multifractal formalism?

- Certainly need $\dim_H \operatorname{supp} \nu = \dim_B \operatorname{supp} \nu$
- measure should also be locally "nice"

3. Self-similar measures

Fix $\mathbb{R}$, finite set of maps $\{S_i\}_{i\in\mathcal{I}}$, each

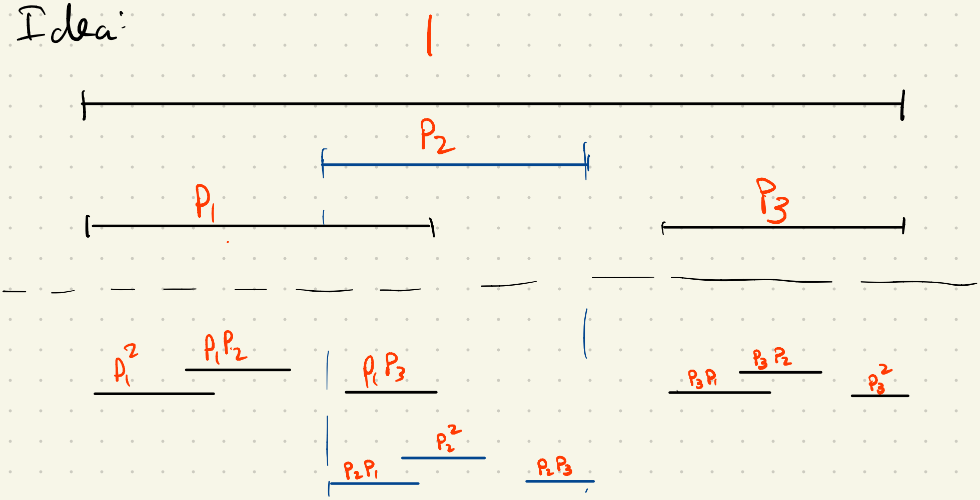
$$S_i(x) = r_i x + d_i \quad \text{with} \quad 0 < |r_i| < 1.$$

Also need probabilities $(p_i)_{i\in\mathcal{I}}$ w/ $p_i > 0$, $\sum p_i = 1$.

Then: $\exists$ unique measure μ s.t.

$$\mu(E) = \sum_{i\in\mathcal{I}} p_i \cdot \mu\left(S_i^{-1}(E)\right).$$

Idea:



- subdivide mass according to probabilities + accumulate overlaps
- Cantor measure from before: $S_1(x) = \frac{x}{3}$, $S_2(x) = \frac{x}{3} + \frac{2}{3}$

$$p_1 = \frac{1}{3} \qquad p_2 = \frac{2}{3}$$

• If ν self-similar, $\dim_B \operatorname{supp} \nu = \dim_H \operatorname{supp} \nu$.

Does multifractal formalism hold for self-similar ν?

> **Theorem:** (Cawley-Mauldin '92)
> $K = \operatorname{supp} \nu$. If $S_i(K) \cap S_j(K) = \emptyset$ for $i \neq j$,
> then ν satisfies multifractal formalism.

In fact, showed also

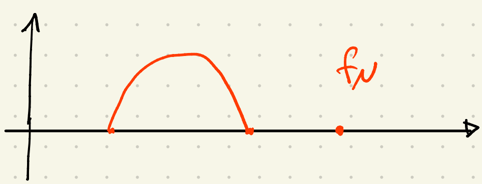
$$\sum_{i \in I} p_i^q \, r_i^{-\tau_\nu(q)} = 1$$

What if overlaps?

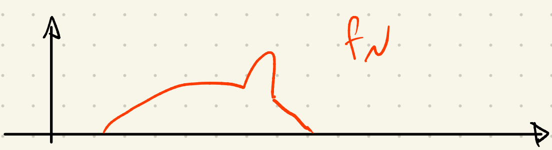
• Example: Hu-Lau '01

If $\upsilon =$ uniform Cantor measure and $\overbrace{\nu = \upsilon * \upsilon * \upsilon}^{\text{self-similar}}$

Then $f_\nu(\alpha)$ is not concave



• Example: Testud '06. Digit-like measures with "very non-concave" spectra



Both examples (in fact, all known examples):

$$f_\nu = \max\{f_1, f_2\}$$

where f_1, f_2 are concave functions.

> Question: Is there a general explanation of this phenomenon?

Suppose ν_1, ν_2 measures with

- ν_1, ν_2 each satisfy multifractal formalism.
- $\operatorname{supp}\nu_1 \cap \operatorname{supp}\nu_2 = \emptyset$

Let $\nu = \nu_1 + \nu_2$. What are $\tau_\nu(q)$ and $f_\nu(q)$?

1. $f_\nu(\alpha) = \dim_H\{x \in \operatorname{supp}\nu : \dim_{loc}(\nu, x) = \alpha\}$

$$= \dim_H\left(\begin{matrix}\{x \in \operatorname{supp}\nu_1 : \dim_{loc}(\nu, x) = \alpha\} \\ \cup \\ \{x \in \operatorname{supp}\nu_2 : \dim_{loc}(\nu, x) = \alpha\}\end{matrix}\right)$$

$\operatorname{supp}\nu_1$ + $\operatorname{supp}\nu_2$ disjoint

$$= \max\{f_{\nu_1}(\alpha), f_{\nu_2}(\alpha)\}$$

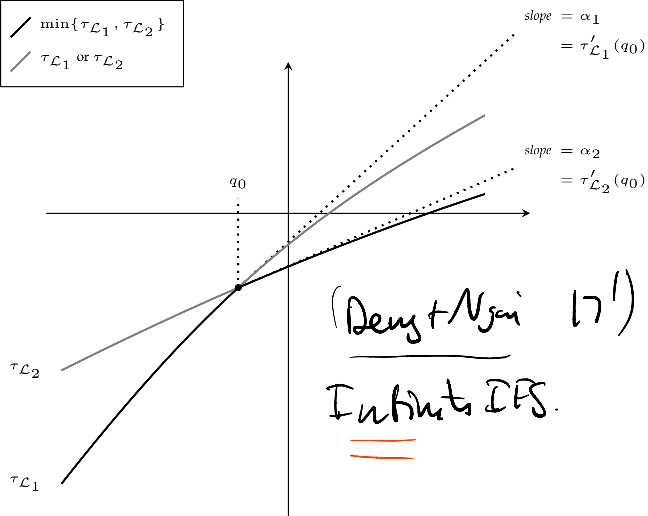
2. $$\tau_\nu(q) = \frac{\log \sup \sum_i \nu(B(x_i, r))^q}{\log r}$$

disjoint support

$$= \frac{\log\left(\sup \sum_{i_1} \nu_1(B(x_{i_1}, r))^q + \sup \sum_{i_2} \nu_2(B(x_i, r))^q\right)}{\log r}$$

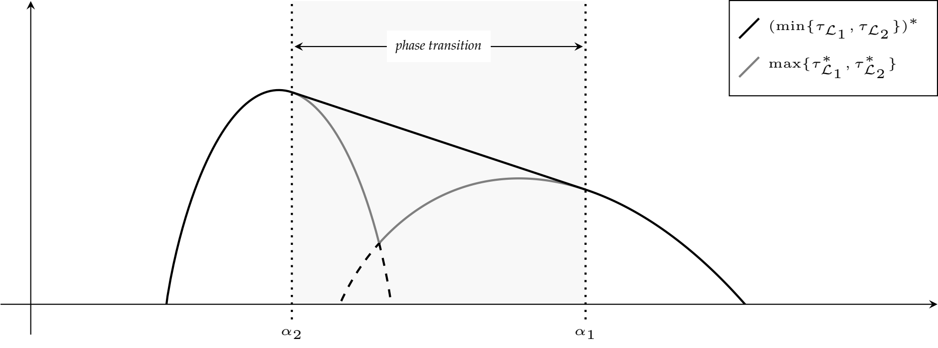
Sum dominated by larger term.

$$= \min\{\tau_{\nu_1}(q), \tau_{\nu_2}(q)\}$$



(Deng + Ngai '17)

Infinite IFS.



phase transition
$(\min\{\tau_{\mathcal{L}_1}, \tau_{\mathcal{L}_2}\})^*$
$\max\{\tau^*_{\mathcal{L}_1}, \tau^*_{\mathcal{L}_2}\}$
α_2
α_1

Theorem: R. '21+ ν self-similar + satisfies weak separation condition + minor technical assumptions. Then $\exists$ concave functions $\tau_1, \ldots, \underline{\underline{\tau_m}}$ s.t.

τ_1 = non-trivial concave $\cap$
τ_2 = linear function

$$\tau_\nu(q) = \min\{\tau_1(q), \ldots, \tau_m(q)\}$$

$$f_\nu(\alpha) = \max\{\tau_1^*(\alpha), \ldots, \tau_m^*(\alpha)\}$$

Corollary: ν satisfies multifractal formalism if and only if f_ν is a concave function.

Contrast to self-affine case, where e.g. for Bedford-McMullen carpets

f_ν is concave, but $f_\nu(\alpha) < \tau^*(\alpha)$

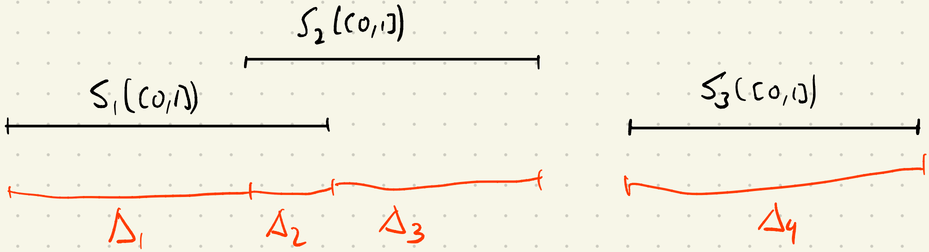
(Jordan-Rams 09').

Reinforces idea that multifractal formalism holds "generically" for self-similar measures

Open Question: If ν <u>any self-similar measure</u>, does f_ν concave $\implies$ ν satisfy multifractal formalism?

Some proof ideas.

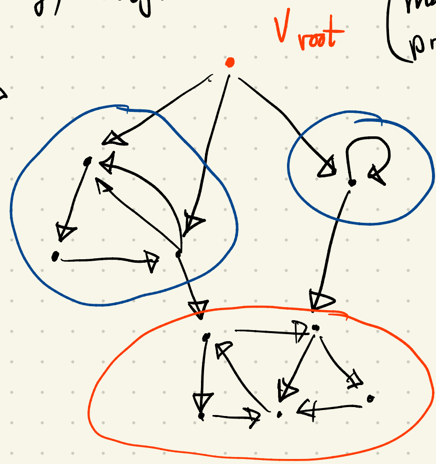
weak separation condition allows for overlaps. E.g.



- Idea: convert overlaps into disjoint behavior
- Construct "graph-directed IFS" on the Δ_i, which are (nearly) disjoint.

(matrices instead of probabilities)

$\rightsquigarrow$



- graph G
- Ω^n = paths of length n
- Ω^∞ = infinite paths

Projection $\Pi_n : \Omega^n \longrightarrow P_n$ = partition of K

induces Lipschitz projection $\Pi : \Omega^\infty \longrightarrow K$

Π "almost injective" (since P_n are partitions)

Get functions

$$T: E(G) \longrightarrow M_d^+(\mathbb{R})$$

positive real-valued matrices

$$W: E(G) \longrightarrow (0,1)$$

s.t. if $\eta = (e_1, \ldots, e_n)$ is a path,

$$\Pi_n(\eta) = \Delta \subseteq \mathbb{R},$$

then

$$\mathrm{diam}(\Delta) = W(\eta) = W(e_1) \cdots W(e_n)$$

$$\nu(\Delta) = \|T(\eta)\| = \|T(e_1) \cdots T(e_n)\|$$

W, T "symbolic" versions of diam, ν.

1. Use correspondence to "lift" multifractal spectrum + L^q spetrum to symbol space Ω^∞
2. Decompose symbol space according to strongly connected components.
3. Prove multifractal formalism for each component.
4. "Stitch together" separate components.