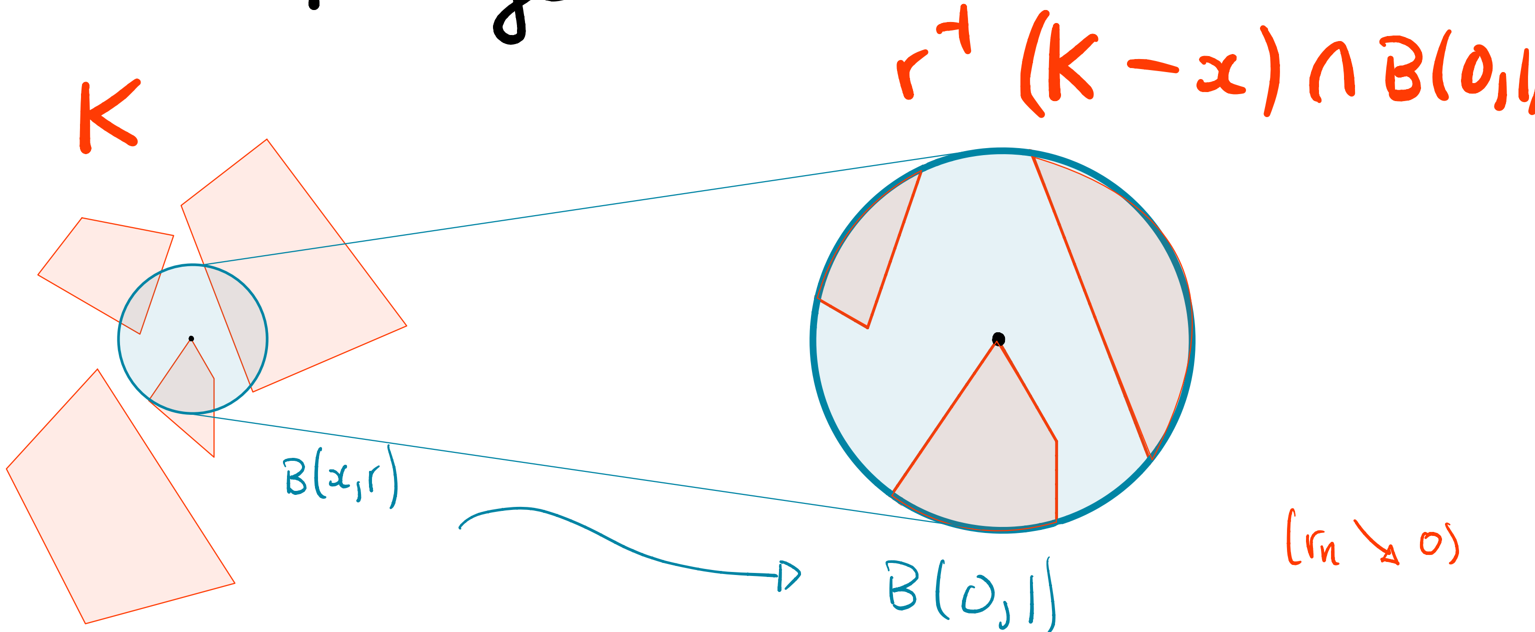
Assouad-type dimensions

{ finer information on scaling and homogeneity }

Alex Rutar // St Andrews

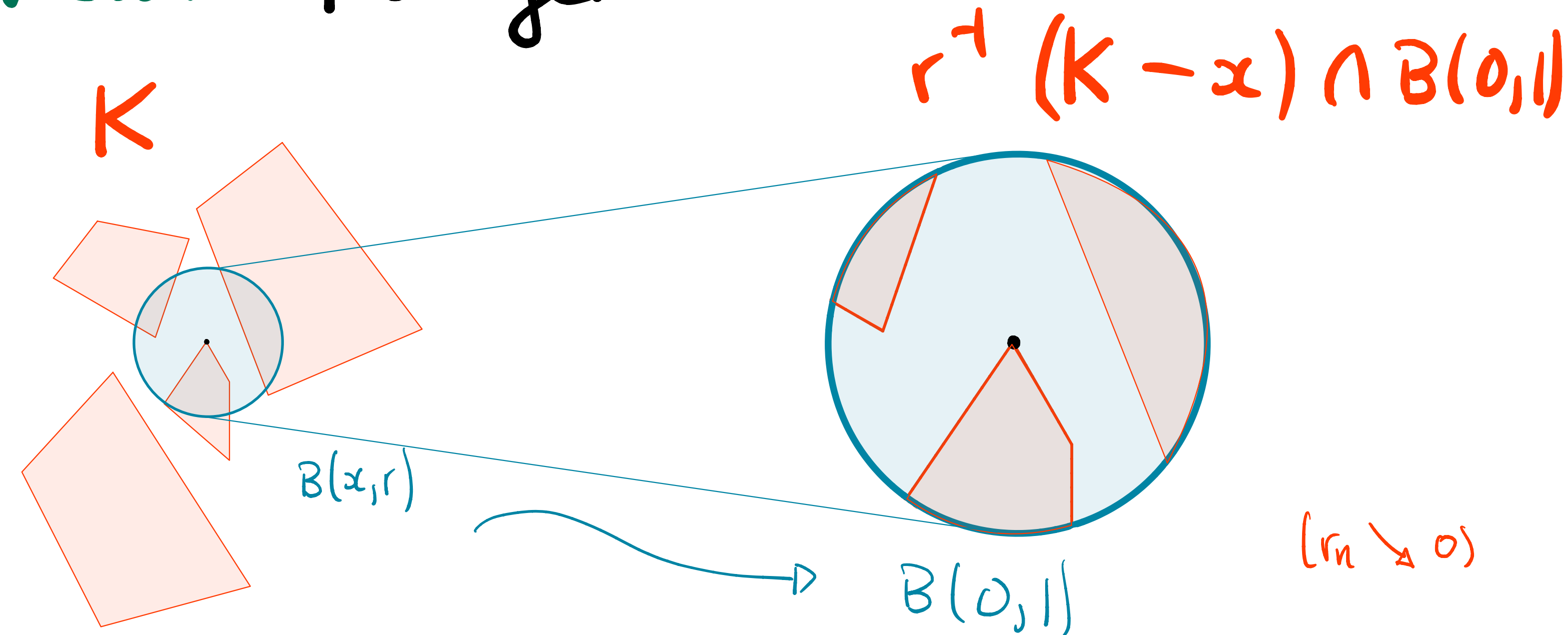
Tangents



Tangent: $\lim\limits_{n\to\infty} r_n^{-1}(K-x)\cap B(0,1)$

(in Hausdorff distance)

Weak Tangents



Weak Tangent: $\lim_{n\to\infty} r_n^{-1}(K-x_n)\cap B(0,1)$

(in Hausdorff distance)

Set of weak tangents: $\mathrm{Tan}(K)$

$$\dim_A K = \sup\{\dim E : E \in \mathrm{Tan}(K)\}$$

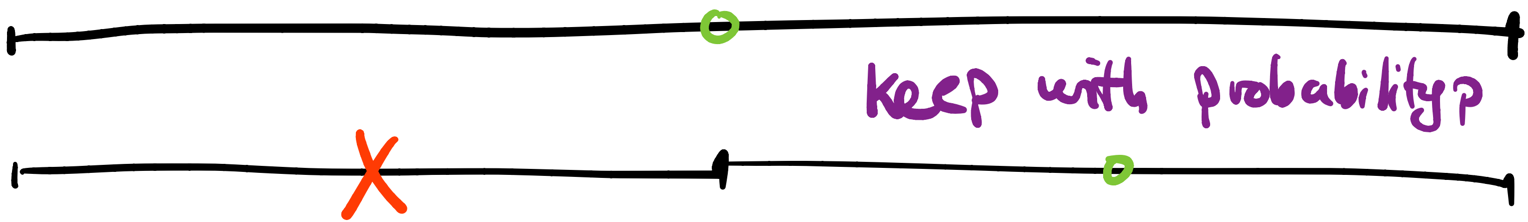
(originally star dimension of Furstenberg, equivalent to Assouad dimension as observed by Käenmäki-Ojala-Rossi)

$$\dim_A K = \inf\{\alpha : \forall\, 0 < r \leq R < 1\ \forall x \in K$$

$$N_r(B(x,R) \cap K) \lesssim \left(\frac{R}{r}\right)^{\alpha}\}$$

Example: Mandelbrot Percolation

fix $p \in [1/2, 1]$



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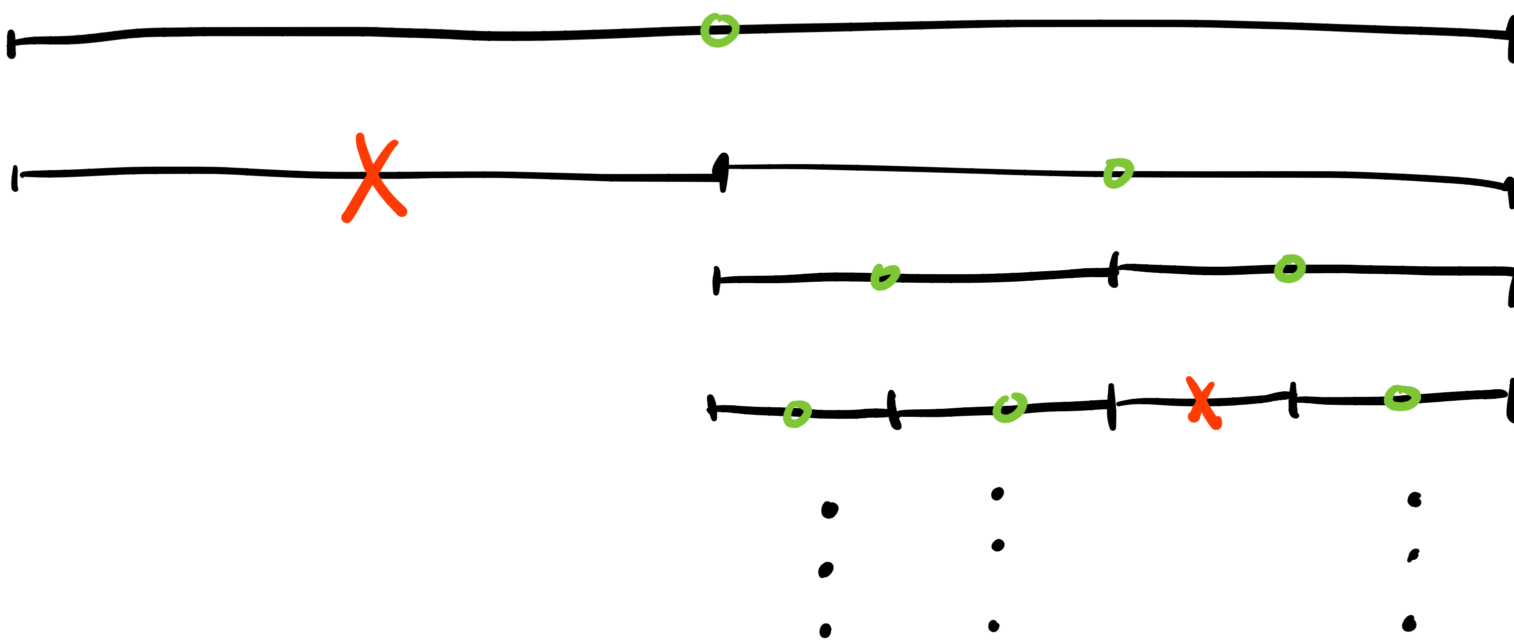
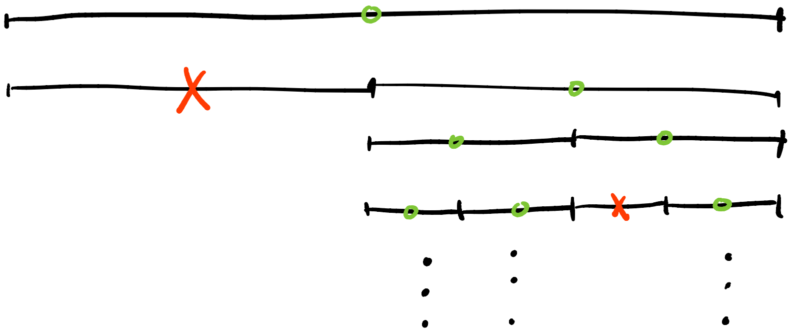
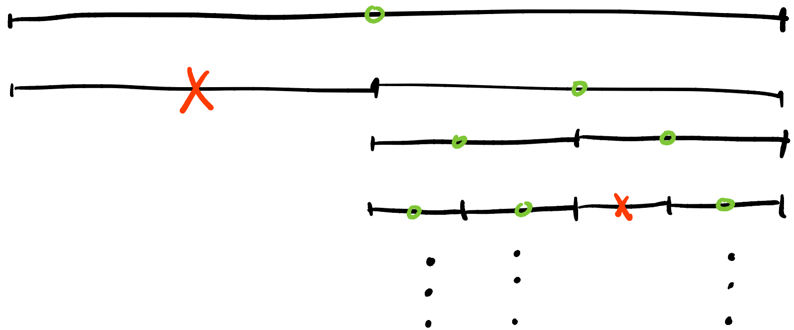




Process gives random set $M_p \subset [0,1]$, non-empty w/ pos. prob.



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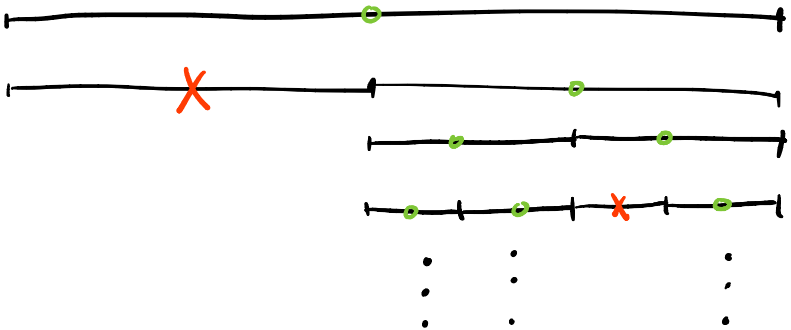
- box dim $= \dfrac{\log(\text{average \# offspring})}{\log(2)} = \dfrac{\log(2p)}{\log 2}$



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HOWEVER: each interval has positive probability of giving full subtree on N levels

$\Longrightarrow$ Assouad dim $= 1$ (independent of p)

Weak tangent has 3 parameters:

1) Location (x) $\qquad \lim\limits_{n\to\infty} r_n^{-1}(K - x_n) \cap B(0,1)$

2) Scale (r)

3) Resolution (level of approximation to limit)

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Quantifying the relationship between these parameters: ϕ-Assouad dimension

Following
- Fraser – Yu (2018)
- García – Hare – Mendivil (2021)
- Banaji – R. – Troscheit (2023+)

Random set example: "large deviations for subtrees"

Example (Banaji-R-Troscheit) Let $\varphi(r) = \dfrac{\log\log(1/r)}{\log(1/r)}$.

Then for $0 < \alpha \leq \log 2$

$$N_{r^{\varphi(r)/\alpha}}\left(r^{-1}(M_p - x) \cap B(0,1) \right)$$

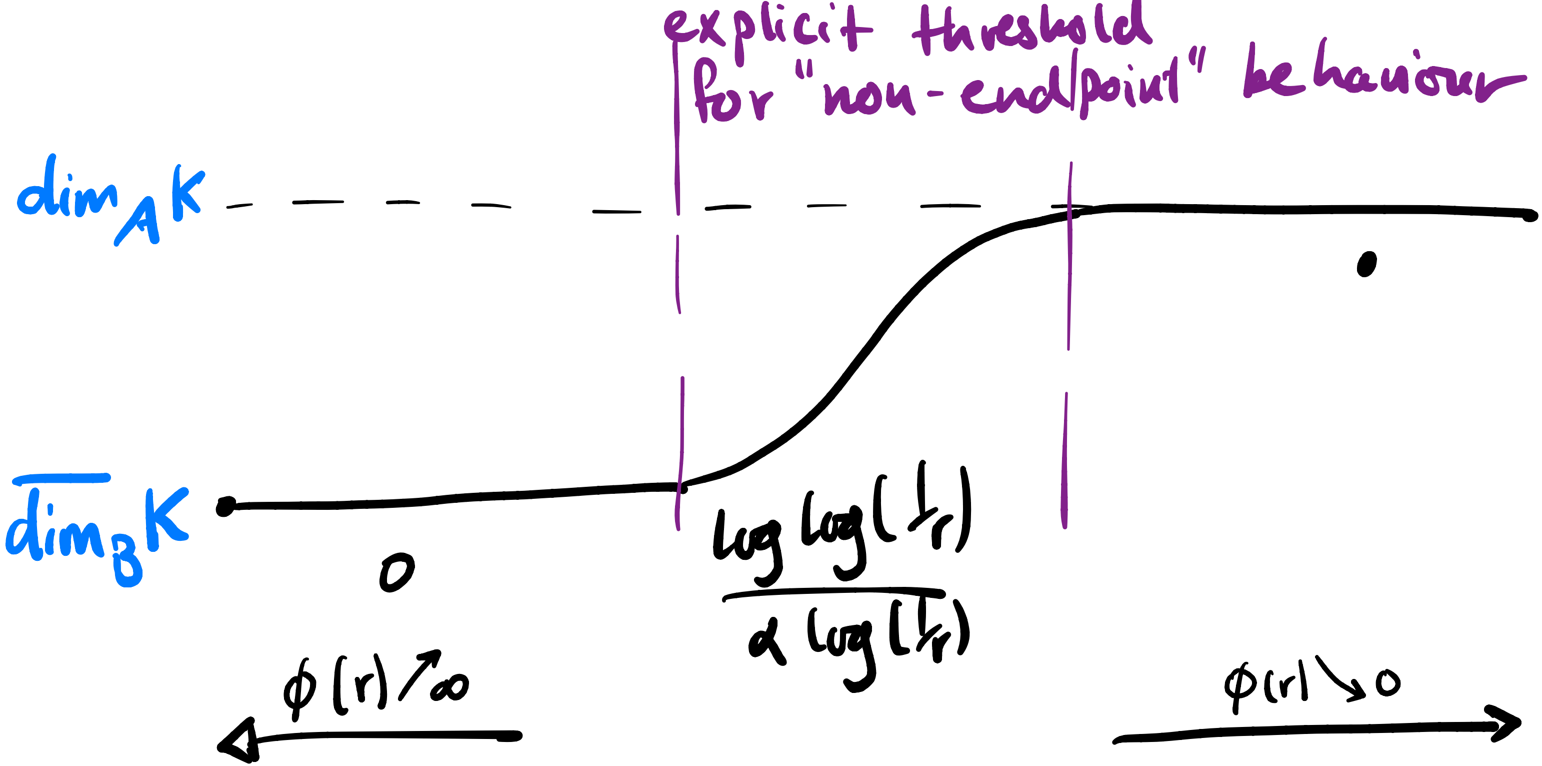
$\dim_B M_p$ at $\alpha = 0$

$=$

$$\approx \left(r^{\varphi(r)/\alpha} \right)^{\frac{\alpha\left(1 - \frac{\log 2p}{\log 2}\right) + \log 2p}{\log 2}}$$

$= \dim_A M_p$ at $\alpha = \log 2$

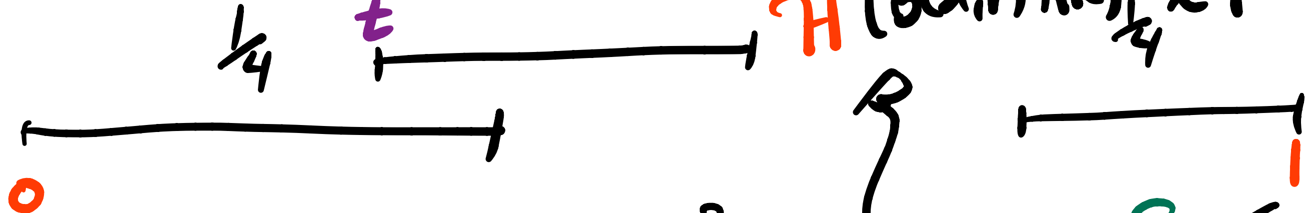
for arbitrarily small r, and exponent is sharp.



domain = mono. decreasing $\phi : (0,1) \to (0,\infty)$

Self-similar sets and pseudo-randomness

$\mathcal{H}^t(B(x,r)\cap K) \underset{1/4}{\approx} r^t$



$\rightsquigarrow$ set C_t, $0<t<\frac{3}{4}$, dimension $\delta_t \leq \frac{\log 3}{\log 4} < 1$

A) $\mathcal{H}^{\delta_t}(C_t) > 0 \iff C_t$ Ahlfors regular

B) $\mathcal{H}^{\delta_t}(C_t) = 0 \iff C_t$ (has weak tangent $[0,1]$)

MOREOVER: A, B occur for parameter sets w/ positive Leb. measure.

Let t be such that $\begin{cases} \delta_t = \frac{\log 3}{\log 4} \\ \mathcal{H}^{\delta_t}(C_t) = 0 \end{cases}$

$= B \setminus (\underbrace{\{\text{countable set}\}}_{})$

Then :

- C_t has weak tangent $[0,1]$
(Fraser - Henderson - Olsen - Robinson, 2016)

- For any $\beta > 1$, any $x \in K$

$$N_{r^\beta}\left(r^{-1}(C_t - x) \cap B(0,1) \right) \lesssim_{\beta,\varepsilon} C \left(r^{\beta} \right)^{\delta_t + \varepsilon} =$$

(Shmerkin, 2019)]

Question: What is threshold function $\psi(r)$ s.t.

$$N_{r^{\psi(r)}}\left(r^{-1}(K-x) \cap B(0,1) \right) \gg \left(r^{\psi(r)} \right)^{-s_t}$$

for inf. many x, r?

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for inf. many x, r?

$B(x,r)$

$$\overline{\dim}_B K < \alpha < \dim_A K \qquad N_{r^{1+\psi(r)}}(B(x,1)\cap K) \approx \left(r^{\psi(r)}\right)^{-\alpha}$$

Theorem (Banaji + R. + Troscheit) Threshold functions always exist for arbitrary bounded subsets of $\mathbb{R}^d$. ("Intermediate value theorem")

Suppose $\varphi(r) = \text{constant} \in (0,\infty)$

Then

$$N_{r^{1+\phi^{\alpha}(r)}}\left(B(x,r) \cap C_t \right) \lesssim \left(r^{\phi^{\alpha}(r)}\right)^{\alpha}$$

$$\underline{=}$$

$\dim_A C_t = 1 \implies$ there must exist function ϕ^{α} for each $\alpha \in (\dim_B K, \dim_A K)$

s.t.

Other Applications

1) Dimension bounds under distortion by {Hölder, quasi-conformal, ...} maps

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   e.g. quasiconformal distortion of spirals (Garitsis – Tyson)

2) L^p-improving properties of spherical maximal functions. $E \subset [1,2]$

   → fully resolved for sets in which the maximal possible scaling appears as early as possible (Roos – Seeger)

Weak tangent has 3 parameters:

1) Location (x) $\lim_{n\to\infty} r_n^{-1}(K - x_n) \cap B(0,1)$

2) Scale (r)

3) Resolution (level of approximation to limit)

Closely related to rectifiability, etc.
but what about for less nice sets?

Problem :

$$K = \{0\} \cup \bigcup_{n=1}^{\infty} \{ 2^{-k} + \ell \cdot 4^{-k} : \ell = 1, \ldots, k \}$$

has weak tangent $[0,1]$

but every tangent has $\leqslant 2$ points.

Problem:

$$K = \bigcup_{n=1}^{\infty} \{ 2^{-k} + \ell \cdot 4^{-k} : \ell = 1, \dots, k \}$$

has weak tangent $[0,1]$

but every tangent has ≤ 2 points.

However, many commonly-studied fractal sets satisfy some form of invariance.

Example. $0 < \mathcal{H}^s(K) < \infty$

Density theorem for Hausdorff content:
for $\mathcal{H}^s$-a.e. $x \in K$,

$$1 \leq \limsup_{r \to 0} \left(\frac{\mathcal{H}^s_\infty(B(x,r) \cap K)}{r^s} \right)$$

$$\underset{\text{sub.}}{=} \lim_{n \to \infty} \mathcal{H}^s_\infty\left(r_n^{-1}(K - x) \cap B(0,1) \right)$$

content is upper-semicts

$$\underset{\text{sub.}}{\leq} \mathcal{H}^s_\infty(E) \leq \mathcal{H}^s(E)$$

$$(E \in \mathrm{Tan}(K,x))$$

$\dim_H E = s$

$\Rightarrow$ Fact: for $\mathcal{H}^s$-a.e. $x \in K$, $\exists\, F \in \mathrm{Tan}(K,x)$ s.t.
$\mathcal{H}^s(F) > 0$.

Good: guarantees <u>many</u> points w/ property

Bad: exponent s not optimal
(want $s = \dim_A K$)

Call a set K **self-embeddable** if

$\forall\ \underline{\underline{x \in K}}\quad \forall\ \underline{\underline{r}} \in (0,1)\ \exists$ bi-Lipschitz

$$f : K \hookrightarrow \underbrace{K \cap B(x,r)}$$

(not necessarily surjective)

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(not necessarily surjective)

→ e.g. any attractor

$$K = \bigcup_{i=1}^{n} f_i(K)$$

$f_i : \mathbb{R}^d \to \mathbb{R}^d$ bi-Lipschitz contractions.

K is **uniformly self-embeddable** if

$\forall x \in K \quad \forall\, 0<r<1 \quad \exists f$

$$f : K \hookrightarrow B(x,r) \cap K$$

$$|f(y) - f(z)| \approx \underline{\underline{r}}\,|x-y|$$

does not depend on x, r

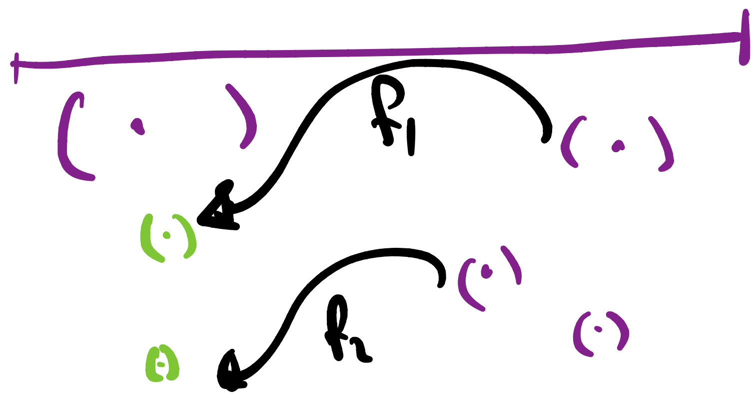
(quantitative control of Lipschitz exponent

e.g. self-similar sets, such as the C_t from earlier.

Theorem (Käenmäki + R.) Let K be self-embeddable. Then

$\exists$ tangent E s.t. $\dim_H E = \dim_A K$.

$E \in \mathrm{Tan}(K, x)$



Theorem (Käenmäki + R.) Let K be (self-embeddable). Then

$\exists$ tangent E s.t. $\dim_H E = \dim_A K$.

Suppose K uniformly self-embeddable. Then

$$\dim_H \left\{ x \in K : \begin{array}{c} (\exists E \in \mathrm{Tan}(K,x)) \\ \dim_H E = \underline{\dim_A} K \end{array} \right\} = \dim_H K$$

$\rightsquigarrow$ optimal exponent + optimal size

Intermediate behaviour?

- examples of self-embeddable sets with

- $\dim_H \left\{ x \in K : \sup\{\dim_H F : F \in \mathrm{Tan}(K,x)\} = \alpha \right\} = \dim_H K$

  for all $\dim_B K \leqslant \alpha \leqslant \dim_A K$

- $\dim_H \left\{ x \in K : \sup\{\dim_H F : F \in \mathrm{Tan}(K,x)\} = \dim_A K \right\} < \dim_H K$