Pisot Numbers and Bernoulli Convolutions

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Algebraic Integers

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- Fact: we can take $p(x)$ *minimal*, i.e. of smallest degree. Such $p(x)$ is unique.

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  - Then $\phi^2 - \phi - 1 = 0$, so ϕ is an algebraic integer with minimal polynomial $x^2 - x - 1$. The other root is $(-\sqrt{5}+1)/2 \approx -0.61803399$.
  - So ϕ is a Pisot number
- Also works for largest real root of $x^k - (x^{k-1} + x^{k-2} + \cdots + 1)$ (Simple Pisot number)

Bernoulli Convolutions

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$$\sum_{n=1}^{\infty} \pm\lambda^n$$
(take $+$ with probability $1/2$ and $-$ with probability $1/2$)
- Distribution has rule μ, i.e.
$$\mathbb{P}(\sum_{n=1}^{\infty} \pm\lambda^n \in A) = \mu(A).$$

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$$\begin{aligned}
&\mathbb{P}\big(\sum_{n=1}^{\infty} \pm\lambda^n \in A\big) \\
&= \frac{1}{2}\mathbb{P}\big(\lambda + \sum_{n=2}^{\infty} \pm\lambda^n \in A\big) + \frac{1}{2}\mathbb{P}\big(-\lambda + \sum_{n=1}^{\infty} \pm\lambda^n \in A\big) \\
&= \frac{1}{2}\mathbb{P}\big(\sum_{n=1}^{\infty} \pm\lambda^n \in A/\lambda - 1\big) + \frac{1}{2}\mathbb{P}\big(\sum_{n=1}^{\infty} \pm\lambda^n \in A/\lambda + 1\big)
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- If the overlaps are "random" would expect μ to be absolutely continuous

Pisot Bernoulli Convolutions

Theorem (Erdős, 1935)

If $\lambda \in (1/2, 1)$ *and* $1/\lambda$ *is Pisot, then* μ *is singular with respect to Lebesgue.*

Proof.

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Let $\theta = 1/\lambda$. Fact: $\mathrm{dist}(\theta^n, \mathbb{Z}) \to 0$ geometrically fast since θ is Pisot. Why? If θ has conjugates $\theta_2, \ldots, \theta_k$, then

$$\theta^n + \sum_{j=2}^{k} \theta_j^n \in \mathbb{Z}$$

(symmetric function of roots). But $\max_{j=2,\ldots,k} |\theta_j| = \rho < 1$ so $\mathrm{dist}(\theta^n, \mathbb{Z}) \leq (k-1)\rho^n$. □

Proof Cont.

Now

$$\hat{\mu}(\pi\theta^N) = \int_{\mathbb{R}} e^{it\pi\theta^N} d\mu(t) = \prod_{n=1}^{\infty} \frac{1}{2} \widehat{(\delta_{\lambda^n} + \delta_{-\lambda^n})}(\pi\theta^N)$$

$$= \prod_{n=1}^{\infty} \cos(\lambda^n \pi\theta^N) = \prod_{n=1}^{N} \cos(\pi\theta^n) \cdot \hat{\mu}(\pi).$$

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But $\pi\theta^n$ converges to an integer multiple of π geometrically fast, so for some $\rho \in (0,1)$

$$|\hat{\mu}(\pi\theta^N)| \geq \prod_{n=1}^{\infty} |\cos(\rho^n)| \cdot |\hat{\mu}(\pi)| \geq \delta > 0$$

for all $N \geq 1$.

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for all $N \geq 1$. Thus $|\hat{\mu}(\xi)| \not\to 0$ as $\xi \to \infty$, so μ is not absolutely continuous by the Riemann-Lebesgue lemma.

Open question

Theorem (Solomyak, Shmerkin)

For all $\lambda \in (1/2, 1)$ *outside an exceptional set with Lebesgue measure 0 (Solomyak) or Hausdorff dimension 0 (Shmerkin), the Bernoulli convolution is absolutely continuous.*

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Open question: what is the exceptional set? Only known counterexamples are reciprocals of Pisot numbers (countable set!)