Some exotic phenomena for Assouad spectra

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Recall: (upper) box dimension $K \subseteq \mathbb{R}^d$; $\mathbb{R}^2$

$$\overline{\dim}_B K = \inf \left\{ \alpha : \exists C > 0 \ \forall\, 0 < r < 1 \quad \underbrace{N_r(K)}_{\#\text{ balls radius } r \text{ cover } K} \leq C \cdot \left(\frac{1}{r}\right)^{\alpha} \right\}$$

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- "average scaling" at scale r, as $r \to 0$
- downsides: often "too coarse", especially for sets with large amounts of inhomogeneity (e.g. self-affine sets) esp. self-affine carpets.

Assouad Spectrum

$\theta \in (0,1)$

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- introduced by Fraser – Yu (2018)
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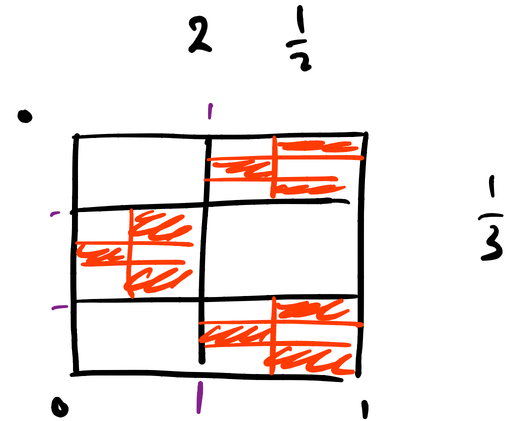
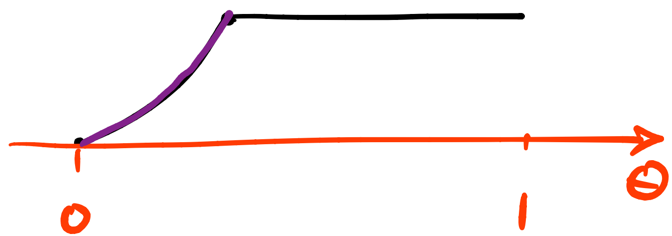
- $\lim_{\theta\to 1} \dim_A^\theta K = \dim_{qA} K$ Fraser–Hare–Hare–Troscheit–Yu (2019)

Examples.

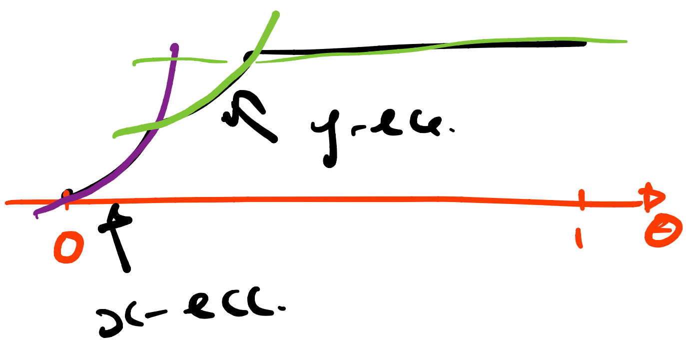
- Bedford-McMullen carpets





(Fraser–Yu)

- elliptical polynomial spirals



(Some) Julia sets, Kleinian limit sets, ...

[$\dim_A^\theta K$ piecewise $a_i + \frac{\theta b_i}{1-\theta}$]

(Some) Julia sets, Kleinian limit sets, ...

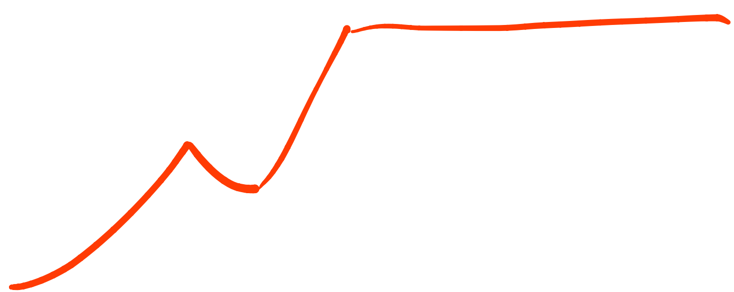
$\dim_A^\theta K$ piecewise $a_i + \frac{\theta b_i}{1-\theta}$

What other behaviour is possible?

Some isolated examples (Fraser-Yu; Fraser-Hare-Hare-Troscheit-Yu)

non-monotonicity

"manual"



pts. of non-diff on every open set

Some isolated examples (Fraser-Yu ;
Fraser-Hare-Hare-Troscheit-Yu)

Theorem (R.) $\exists$ bounded set $\emptyset \neq F \subset \mathbb{R}^d$ s.t.

$$\dim_A^\theta F = \varphi(\theta)$$

$$\Longleftrightarrow$$

$\varphi : (0,1) \to [0,d]$ and $\forall\ 0 < \lambda < \theta < 1$

$$0 \leq (1-\lambda)\varphi(\lambda) - (1-\theta)\varphi(\theta) \leq (\theta-\lambda)\varphi\left(\frac{\lambda}{\theta}\right)$$

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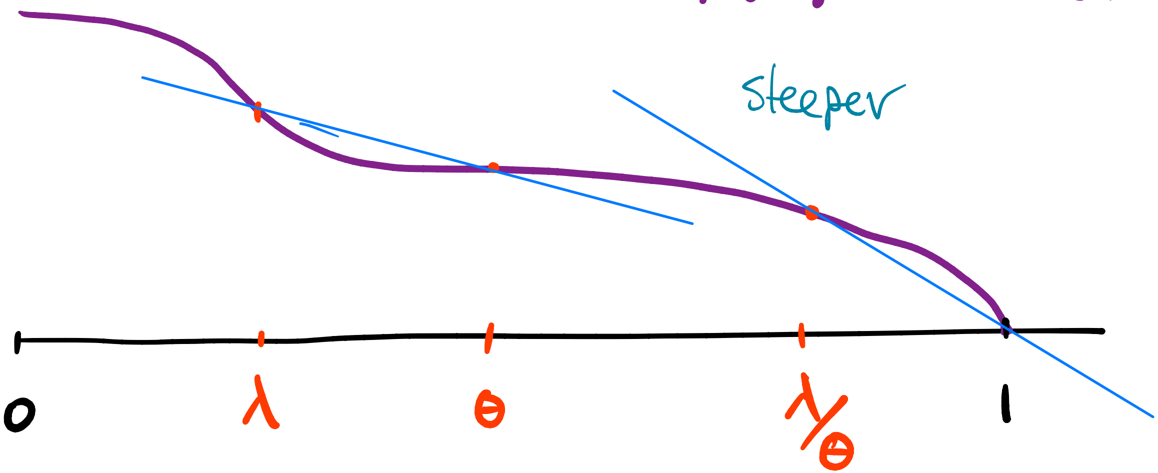
β decreasing

"osscilation" condition

$$\beta(\theta) = (1-\theta)\cdot\varphi(\theta)$$

steeper



$\Rightarrow$ follows by direct covering argument;
already in Fraser–Yu (2018)

$\Leftarrow$ Via "homogeneous Moran set construction"
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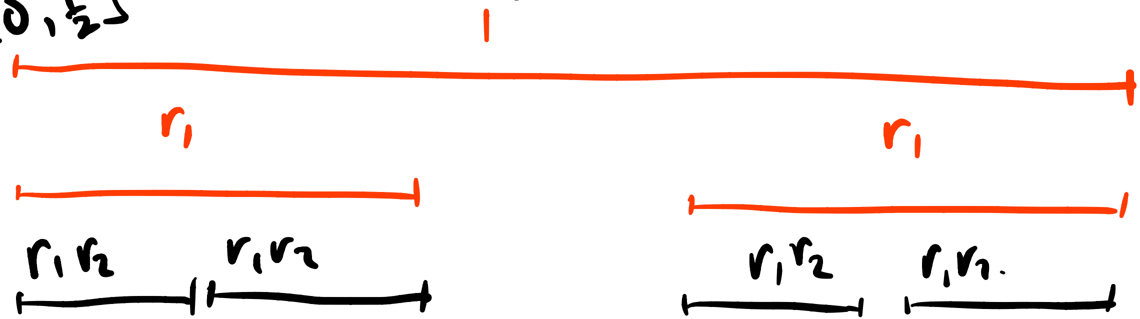
$(r_i)_{i=1}^{\infty} \subset [0, \frac{1}{2}]$



$C((r_i)_{i=1}^{\infty})$

$\Rightarrow$ follows by direct covering argument; already in Fraser–Yu (2018)

$\Leftarrow$ Via "homogeneous Moran set construction" based on techniques from Banaji–Rutar (2022).

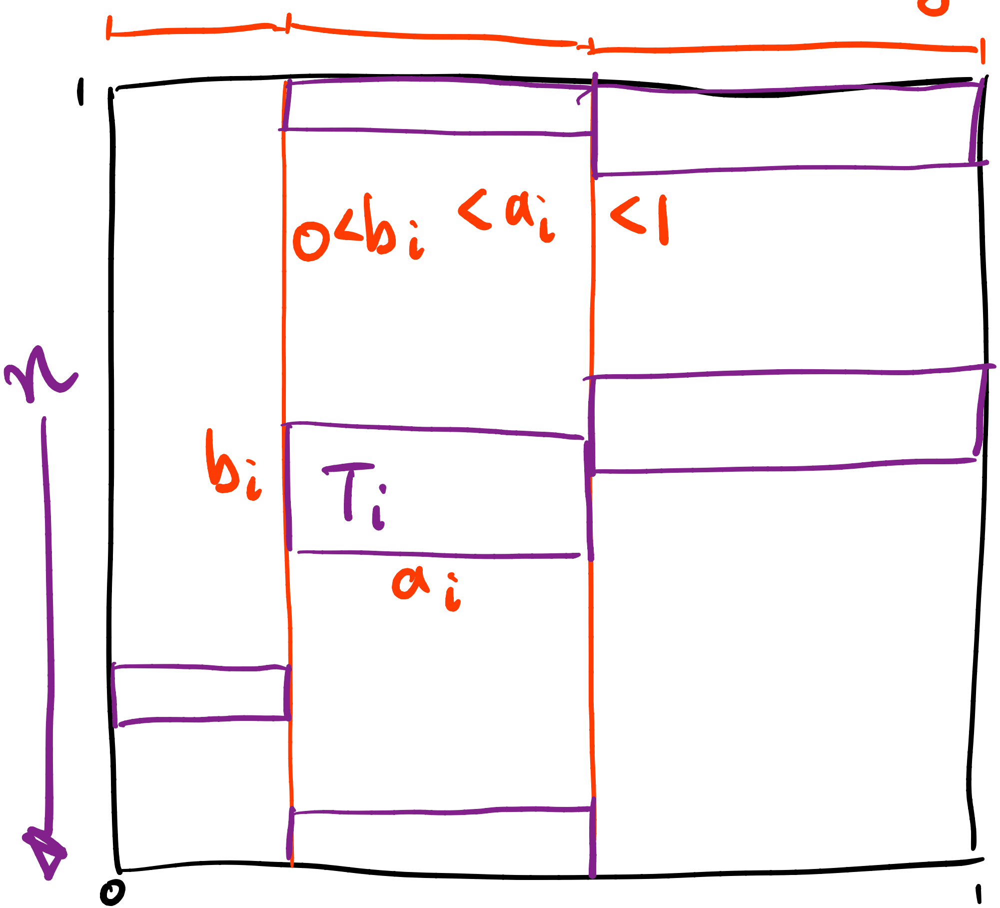
Consequences: $\dim_A^\theta F$ can be:

- non-monotonic on every open set.
- not Hölder at 1.

Construction in classification is <u>very much</u>

non-dynamical.

Gatzouras – Lalley Carpets



$T_i(\square) =$ ▭

$\{T_i\}_{i \in \mathcal{I}}$ strictly contracting. ⇝ K attractor

- $T_i(x,y) = (a_i x, b_i y) + \_\_\_\_$
- $\pi(x,y) = x$

$$t_{min} = \dim_B K - \dim_B \pi(K)$$ "average fibre dim"

$$t_{max} = \underline{\dim_A K} - \underline{\dim_B \pi(K)}$$ <u>maximal</u> fibre dim

$$= \max_{x \in \pi(K)} \dim_B \pi^{-1}(x) \cap K$$

vert. line thru x

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$\pi(\mathcal{I})$ $\quad \pi : \mathcal{I} \longrightarrow$ "columns"

$\pi(\mathcal{I})$ = column indices. For column $\underline{j} \in \pi(\mathcal{I})$ define

$$\psi_{\underline{j}}(t) = \frac{\log \sum_{i \in \pi^{-1}(\underline{j})} \underline{\underline{b}}_i^t}{\log a_{\underline{j}}}$$

indices in col.

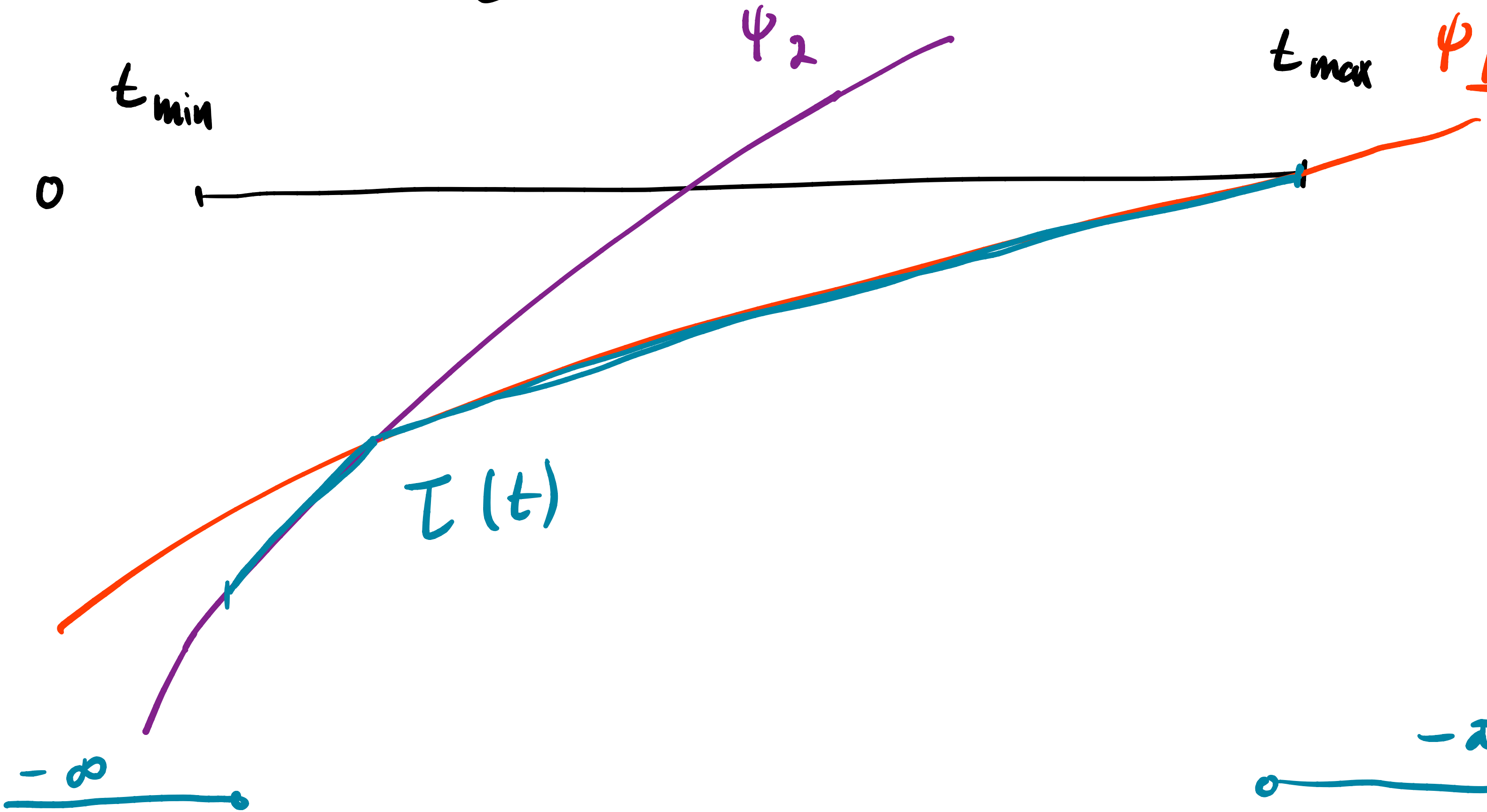
b_i = height of rect

$a_{\underline{j}}$ = width of col.

w/ column.

"column pressure function"

$$\cdot\ \tau(t) = \begin{cases} \min\limits_{\underline{j}} \psi_{\underline{j}}(t) & : t_{min} \leqslant \underline{\underline{t}} \leqslant t_{max} \\ -\infty & : \text{otherwise.} \end{cases}$$



Change of parameter:

$$\phi(\theta) = \frac{1/\theta - 1}{1 - 1/\kappa_{max}} \quad ; \quad \kappa_{max} = \max_{i \in \mathcal{I}} \overbrace{\frac{\log b_i}{\log a_i}}^{\text{log. eccentricity}} \in (1, \infty)$$

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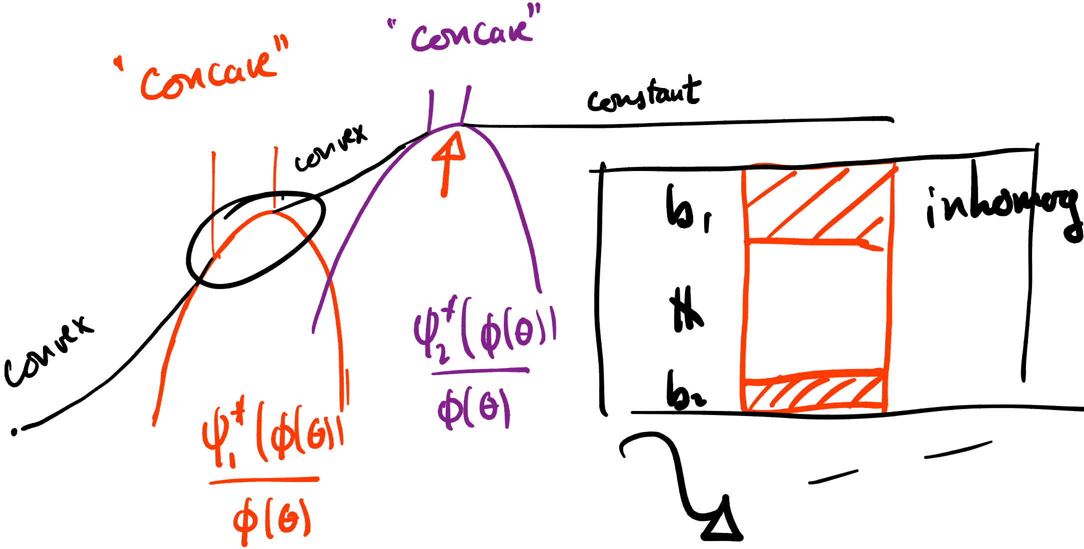
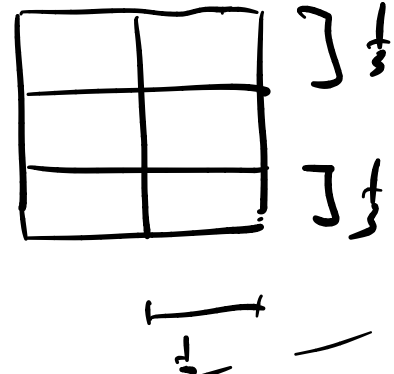
Theorem (Banaji–Fraser–Kolossváry–R.) Let K Gatzouras–Lalley carpet

$$\dim_A^\theta K = \dim_B \pi(K) + \frac{\tau^*(\phi(\theta))}{\phi(\theta)}$$

$$\left[\tau^*(\alpha) = \inf_{t \in \mathbb{R}} (t\alpha - \tau(t)) \; ; \; \text{concave conjugate}\right]$$


"concave"
"concave"
convex
constant
convex
$\frac{\varphi_1^{\dagger}(\phi(\theta))}{\phi(\theta)}$
$\frac{\varphi_2^{\dagger}(\phi(\theta))}{\phi(\theta)}$
b_1
b_2
inhomog


$\frac{1}{3}$
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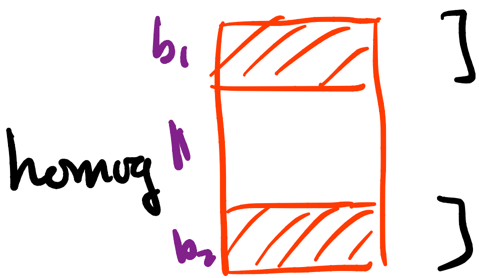
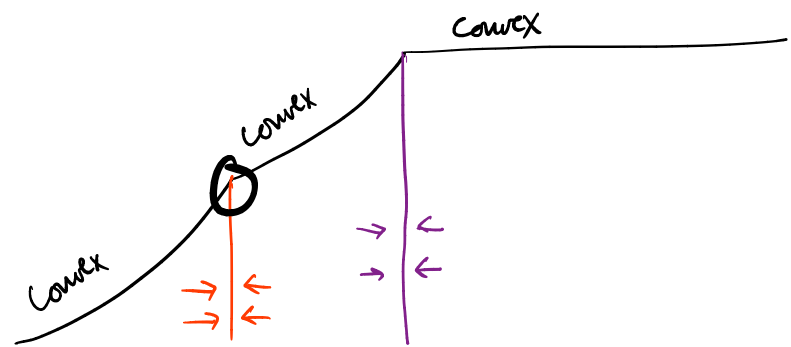

b_1
b_2
homog


convex
convex
convex

Features:

generic in param space

- (can be) differentiable on (0,1)
- strictly concave on non-trivial interval

- piecewise analytic

φ

- phase transitions (order odd integer or 2)

(1, 3, 5, 7, 9, ...) → taylor's other

2

Some tools from proof.

- method of types (large deviations)
- non-convex, non-differentiable optimization theory.
  (parametric geometry of Lagrange multipliers; R., 2023+; probably known earlier?)

Proof Sketch.

$\mathcal{S} = \bigcup \mathcal{S}_i$ → strip height uniform

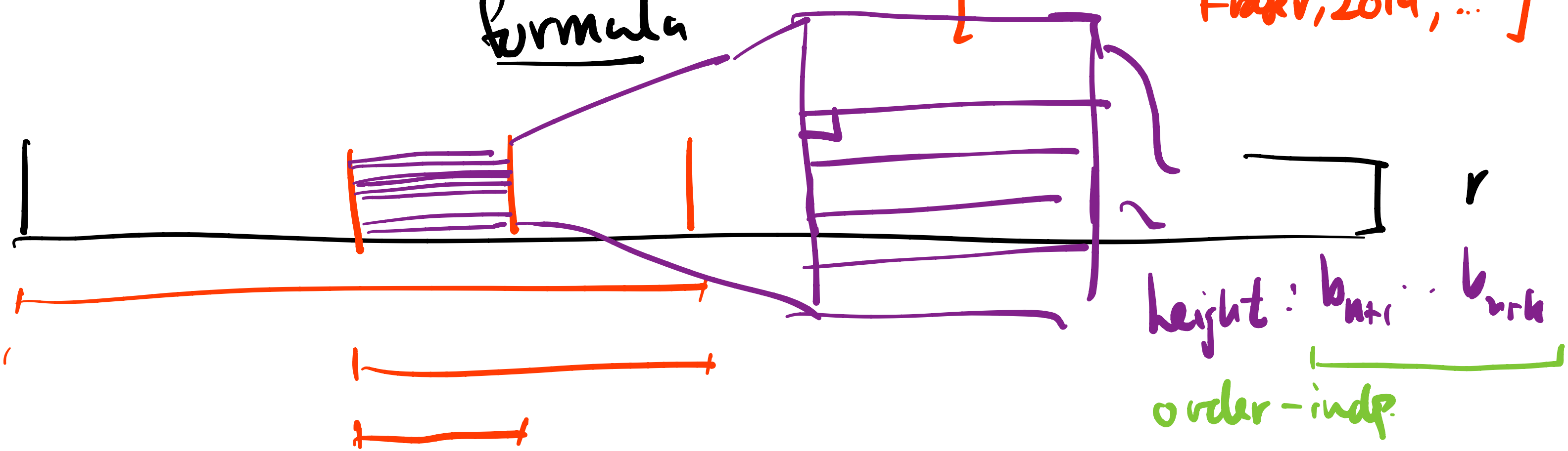
small union

(1) Use method of types + geometric covering arguments

[large deviations]

[following Käenmäki-R., 2023+; Fraser, 2014, ...]

to prove variational formula



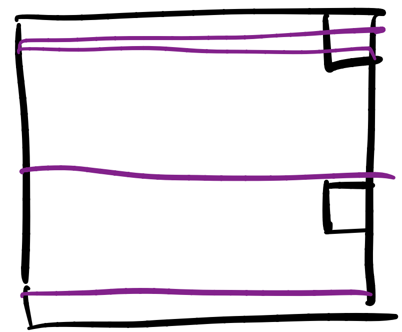
vector space

$$f(\theta, \underline{v}, \underline{w}) = \begin{cases} f_1(\theta, \underline{v}, \underline{w}) : (\underline{v}, \underline{w}) \in \Delta_1(\theta) \\ f_2(\theta, \underline{v}, \underline{w}) : (\underline{v}, \underline{w}) \in \Delta_2(\theta) \end{cases}$$

linear constraint

f_1, f_2 smooth, non-convex; NOT smooth on bdry

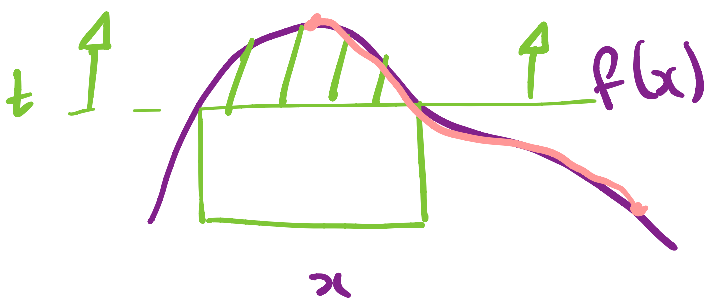
$$\dim_A^{\theta} K = \max_{(\underline{v}, \underline{w})} f(\theta, \underline{v}, \underline{w}) \qquad r^{t_\theta}$$



r

(2) Solve optimization using

- Parametric geometry of Lagrange multipliers (R. 2023+, developed for multifractal analysis)
- information theory / entropy arguments
- "abstract gradient ascent argument"



$\{x : f(x) \geq t\}$

connected.