Multifractal Formalism for Self-Similar Measures with Exact Overlaps

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A Brief Introduction to Multifractal Analysis for Self-Similar Measures

Quantifying Singularity: Local Dimensions

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- x is an atom of μ: $\dim_{\mathrm{loc}}\mu(x) = 0$.

Quantifying Singularity: Multifractal Spectrum

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$$f_\mu(\alpha) = \dim_H\{x \in \operatorname{supp}\mu : \dim_{\mathrm{loc}}(\mu, x) = \alpha\}$$

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- f_μ "measures singularity" for measure μ with respect to Lebesgue measure.
- *multifractal analysis*: determine or understand properties of measures when f_μ has non-trivial domain

L^q-spectrum

Another measure of singularity:

- *L^q-spectrum* of μ, for $q \in \mathbb{R}$,

$$\tau_\mu(q) := \liminf_{t \to 0} \frac{\log \sup \sum_i \mu(B(x_i, t))^q}{\log t},$$

supremum over disjoint families $\{B(x_i, t)\}$, $x_i \in \operatorname{supp} \mu$

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$$\frac{t^q}{t}$$

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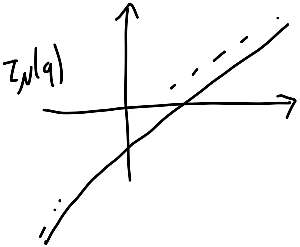
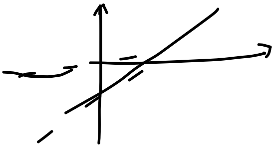
- If μ is Lebesgue on $[0,1]$, then $\tau_\mu(q) = q - 1$.

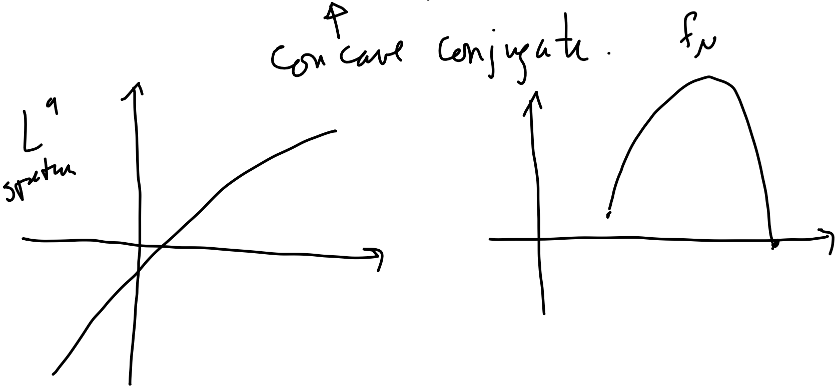




Multifractal formalism

Heuristic relationship (Halsey et al. 1986): if μ is “sufficiently regular”, f_μ is concave and

$$f_\mu(\alpha) = \tau_\mu^*(\alpha) := \inf_{q\in\mathbb{R}} \{q\alpha - \tau_\mu(q)\}$$



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If true, μ satisfies the *multifractal formalism*

Self-Similar Measures

- *iterated function system of similarities* (IFS): $\{S_i\}_{i=1}^m$, $S_i : \mathbb{R} \to \mathbb{R}$,

$$S_i(x) = r_i x + d_i \qquad 0 < |r_i| < 1.$$

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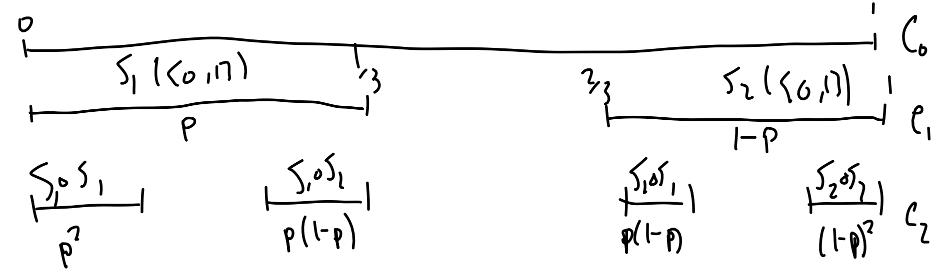
$$K = \bigcup_{i=1}^{m} S_i(K)$$

- Given $(p_i)_{i=1}^m$, $p_i > 0$, $\sum_{i=1}^m p_i = 1$, unique (Borel probability) *self-similar measure* μ:

$$\mu(E) = \sum_{i=1}^{m} p_i \mu(S_i^{-1}(E)) \text{ for } E \subseteq K = \operatorname{supp} \mu.$$

Cantor Set and Measure

$\left\{ \frac{x}{3}, \ \frac{x}{3} + \frac{2}{3} \right\}$ $\qquad p, \qquad 1-p$



$\vdots$

$\mathcal{C} = \bigcap_{n=0}^{\infty} C_i$ etc.

Strong Separation Condition and the Multifractal Formalism

An IFS satisfies:

- the *strong separation condition* if

$$S_i(K) \cap S_j(K) = \emptyset \text{ whenever } i \neq j$$

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- the *open set condition* if there is some V such that

$$\biguplus_{i=1}^{k} S_i(V) \subseteq V \text{ disjointly}$$

  (for example, the IFS $x/2, x/2 + 1/2$).

Multifractal formalism for Open Separation Condition

Theorem (Cawley, Mauldin 1992; Patzschke 1997; Pesin, Weiss 1997)

μ satisfies the open set condition: multifractal formalism holds where $\tau(q)$ is analytic and satisfies

$q = 0$

$$\sum_{i=1}^{m} p_i^q r_i^{-\tau(q)} = 1.$$

Cawley and Mauldin: true when μ satisfies the strong separation condition.

$$\sum_{i=1}^{m} r_i^s = 1 \qquad s = \underline{\dim}_B K$$

IFSs with Overlaps

For $\sigma = (i_1, \ldots, i_n) \in \{1, \ldots, m\}^n$, write

$$S_\sigma = S_{i_1} \circ \cdots \circ S_{i_n} \qquad r_\sigma = r_{i_1} \circ \cdots \circ r_{i_n}$$

Definition (Bandt, Graf 1992; Lau, Ngai 1999)

The IFS satisfies the *weak separation condition* (WSC) if

$$\sup_{x \in \mathbb{R}, t > 0} \#\{S_\sigma : r_\sigma \approx t, S_\sigma(K) \cap U(x,t) \neq \emptyset\} < \infty. \tag{1}$$

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Note that WSC allows for **exact overlaps**; WSC with no exact overlaps is open set condition

Convolutions of Cantor Measure

- Let $\nu =$ Cantor measure with probabilities $p_1 = p_2 = 1/2$: then

$$\mu = \underbrace{\nu * \cdots * \nu}_{m \text{ times}}$$

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- Satisfies OSC for $m = 1, 2$.
- (Hu, Lau 2001) For $m \geq 3$, there is an isolated point in the set of local dimensions, so the multifractal formalism fails!
- In some sense, the measure is too small on the endpoints of $[0, 1]$ which contributes the isolated values (multifractal formalism fails for $q < 0$)

Bernoulli Convolutions with Pisot Contractions

- (Feng, 2016) IFS $\{rx, rx + (1 - r)\}$ satisfies the WSC if and only if $1/r$ is *Pisot* (real algebraic integer > 1 with Galois conjugates having modulus < 1)

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- (Feng, 2005) When $r = (\sqrt{5} - 1)/2$, μ satisfies multifractal formalism for any probabilities, but $\tau_\mu(q)$ is not differentiable at some $q < 0$

Other Examples

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$$S_1(x) = \rho x \quad S_2(x) = rx + \rho(1 - r) \quad S_3(x) = rx + (1 - r)$$

where $\rho + 2r - r\rho \leq 1$.

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  where $\rho + 2r - r\rho \leq 1$.
- (Deng, Ngai 2017) Modified multifractal formalism with respect to differentiable auxiliary function $\tilde{\tau}$ for probabilities $p_2 > p_3$.

Theorem (Feng, Lau 2009)

Let μ be a self-similar measure satisfying the WSC. Then μ satisfies the multifractal formalism for $q \geq 0$. Moreover, there exists an open set U with $\mu(U) > 0$ such that $\mu|_U$ satisfies the multifractal formalism for all $q \in \mathbb{R}$.

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- U is any open ball which attains maximality in the definition of the WSC.
- μ is "more regular" on U (avoids the "small-weight" problems for $q < 0$)

New Results: The Transition Graph and Relationships with the Multifractal Formalism

Self-similar measures as projections

Fix IFS $\{S_i\}_{i=1}^m$.

- Let $\Sigma = \{1,\ldots,m\}^{\mathbb{N}}$ and define $\pi : \Sigma \to K$ by

$$\pi(i_1, i_2, \ldots) = \lim_{k\to\infty} S_{i_1} \circ \cdots \circ S_{i_k}(0)$$

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- strong separation condition: π injective and $\mu(S_{i_1} \circ \cdots \circ S_{i_n}(K)) = p_{i_1} \cdots p_{i_n}$
- What happens when π is not injective (the pieces overlap)?

Decomposing Overlaps: Neighbour Sets

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- $\Delta \in \mathcal{F}_t$ (called a *net interval*): define the *neighbour set* of Δ

$$\mathcal{V}_t(\Delta) = \mathcal{V}(\Delta) = \{T_\Delta^{-1} \circ S_\sigma : r_\sigma \approx t, S_\sigma(K) \cap (0,1) \neq \emptyset\}$$

where $T_\Delta(x) = rx + d$ with $r > 0$ has $T_\Delta([0,1]) = \Delta$.

Neighbour Set Example

Neighbour Set Characterization

Lemma (R. 2020?)

Suppose Δ_1, Δ_2 are net intervals with $\mathcal{V}(\Delta_1) = \mathcal{V}(\Delta_2)$. Then there exists a surjective similarity $g : \Delta_1 \cap K \to \Delta_2 \cap K$ such that if $E \subseteq \Delta_1$ is any Borel set,

$$\mu(g(E)) \approx \mu(E).$$

Idea: let $g = T_{\Delta_1}^{-1} \circ T_{\Delta_2}$:

$$\begin{aligned} g(\Delta_1 \cap K) &= \bigcup_{f \in \mathcal{V}(\Delta_1)} \left(g \circ T_{\Delta_1} \circ f(K)\right) \cap g(\Delta_1) \\ &= \bigcup_{f \in \mathcal{V}(\Delta_2)} \left(T_{\Delta_2} \circ f(K)\right) \cap \Delta_2 = \Delta_2 \cap K. \end{aligned}$$

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- Result: construct *transition graph* G with vertex set $V(G) = \{\text{all neighbour sets}\}$, root $\mathcal{V}([0,1])$, and (directed) edge set

$$E(G) = \{(\mathcal{V}(\Delta), \mathcal{V}(\Delta')) : \Delta \text{ a child of } \Delta'\}$$

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$$E(G) = \{(\mathcal{V}(\Delta), \mathcal{V}(\Delta')) : \Delta \text{ a child of } \Delta'\}$$

- This is the "quotient" of the natural tree-structure of net intervals (under inclusion) where net intervals are equivalent if they have the same neighbour set

Tansition Graph Continued

- Natural bijection

$$\Delta : \{\text{finite rooted paths}\} \to \{\text{net intervals}\}$$

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$$\Delta : \{\text{finite rooted paths}\} \to \{\text{net intervals}\}$$

- Natural "almost injective" (injective outside countable set, and fibres of size at most 2 otherwise) projection

$$\pi : \{\text{infinite rooted paths}\} \to K$$

by $\{\pi(e_1, e_2, \ldots)\} = \bigcap_{n=1}^{\infty} \Delta(e_1, \ldots, e_n)$

Relationship to WSC

Standing Assumption

Now assume there are only finitely many neighbour sets

- equivalently the transition graph is finite

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Theorem (Feng 2016; Hare, Hare, R. 2020?)

Suppose $K = [0, 1]$. Then the IFS satisfies the weak separation condition if and only if there are only finitely many neighbour sets.

Feng proved this for IFS of the form $\{rx + d_i\}_{i=1}^{m}$

Properties of Transition Graph

Let $\Delta \leftrightarrow (e_1, \ldots, e_n)$.

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- There exist *edge weights* $W(e) \in (0,1)$ for each $e \in E(G)$ such that
$$\mathrm{diam}(\Delta) = W(e_1) \cdots W(e_n)$$
- There exist non-negative *transition matrices* $T(e)$ for each $e \in E(G)$, such that
$$\mu(\Delta) = \| T(e_1) \cdots T(e_n) \|_1$$

Symbolic Encoding vs. Transition Graph Encoding

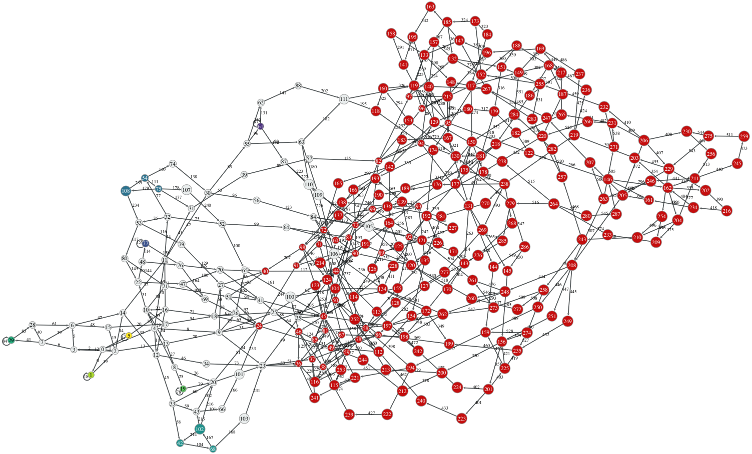
Symbolic Encoding	Transition Graph Encoding
(finite) sequences in $\{1, \ldots, m\}$	(finite) rooted paths in G
$\lim_k S_{i_1} \circ \cdots \circ S_{i_k}(0)$	$\bigcap_{k=1}^{\infty} \Delta(e_1, \ldots, e_k)$
contraction ratios r_i	edge weights $W(e)$
probabilities p_i	transition matrices $T(e)$
strong separation: injective projection	any IFS: almost-injective projection

Essential Class of the Transition Graph

Theorem (R. 2020?)

Under the WSC, the subgraph G has a unique attractor G_{ess} which has "strong measure regularity properties", and such that the corresponding points in $K_{\text{ess}} \subseteq K$ has $\mu(K \setminus K_{\text{ess}}) = 0$.

- The attractor has the property that for any vertex v, there is a path from v to any vertex in the attractor, and any (directed) path starting in the attractor must also end in the attractor



A Modified Multifractal Formalism

Theorem (R. 2020?)

Let μ satisfy the WSC. There exists a sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots \subseteq K$ such that $\lim_i \mu(K_i) = 1$ and each $\mu|_{K_i}$ satisfies the multifractal formalism (with the same L^q-spectrum and fine multifractal spectrum).

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- Here, the sets K_i are "level i approximations of K_{ess}".
- The multifractal formalism follows from the "measure regularity properties" in the same spirit as Feng, Lau 2009.

Combinatorial Perspective on the Multifractal Formalism

Theorem (R. 2020?)

Let μ satisfy the WSC. Suppose in addition that G is a finite graph (this holds, for example, when $K = [0, 1]$) and that there is no cycle in G outside G_{ess}. Then $K_{\text{ess}} = K$ and μ satisfies the multifractal formalism.

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- G finite, no cycle in G outside G_{ess}: get M such that any path of length at least M terminates in G_{ess}

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Combinatorial Perspective on the Multifractal Formalism

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- Recall the bijection $\{\text{net intervals}\} \leftrightarrow \{\text{finite paths in } G\}$
- Since every finite path of length M is in the essential class, the "level M approximation" K_M is actually equal to K.

Example Application

Corollary

Any measure associated with the IFS

$$S_1(x) = \rho x \qquad S_2(x) = rx + \rho(1-r) \qquad S_3(x) = rx + (1-r)$$

satisfies the multifractal formalism. Combining this with (Deng, Ngai 2017), the L^q-spectrum is differentiable whenever $p_2 > p_3$.

Compute $G = G_{ess}$

Weight vs. Combinatorial Connection Intuition

Questions / Extensions

- "Characterization" only goes one way (multifractal formalism fails implies loop outside G_{ess}). Can we do better?
- Can we also get information from systems with loops outside the essential class? (work in preparation with Kathryn Hare)
- Is the transition graph useful outside WSC?
- Higher dimensions?
- Other types of systems?