Analysis Group Intro Talk

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1. Fractals and Dimension Theory

2. Dynamical Systems

Fractals and Dimension Theory

Fractals in Nature?

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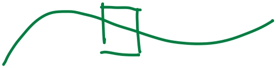
Fractals?

What properties do fractals have?

Fractals?



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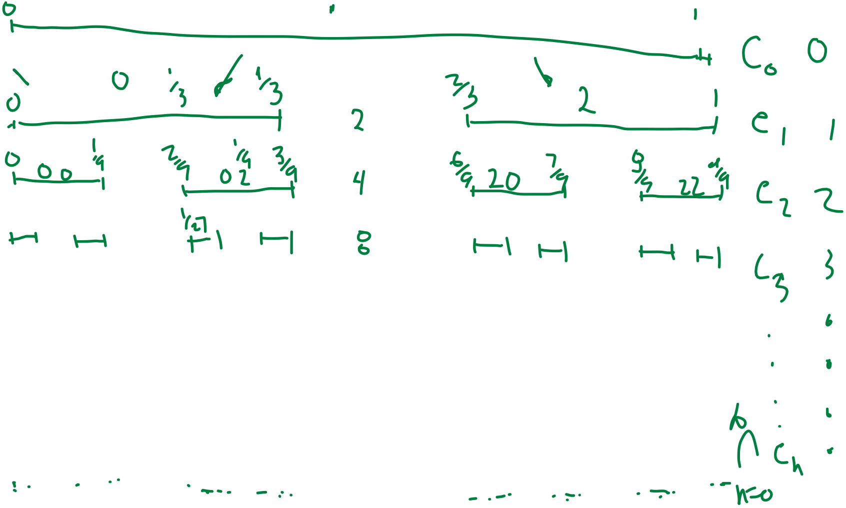
But...no clear definition (most sensible attempts at definitions have exceptions)!

Cantor Set



0
1
C_0
0
1/3
2/3
2
C_1
1
1/9
2/9
3/9
6/9
7/9
8/9
00
02
20
22
4
C_2
2
8
C_3
3
$\bigcap_{n=0}^{\infty} C_n$

Cantor Set

$$x = 0.d_1 d_2 d_3$$

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Cantor Set

$0.b_1 b_2 b_3$ etc.

$0.01101 \to 0.02202$

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  - 'bijection' taking ternary representations to binary representations

Iterated Function System

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- Koch curve
- Starting shape does not matter!

Koch Curves

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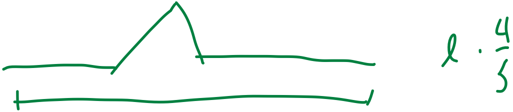

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Koch Curves

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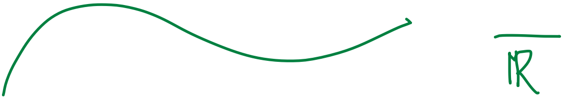
- how to distinguish the different curves?

→ dimension theory

Dimension



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- one idea: dimension as ‘scaling property’

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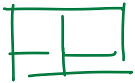

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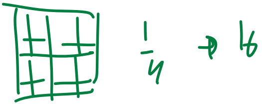

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This is known as the Box / Minkowski dimension

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Example: dimension distinguishes the general Koch curves

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- Many other types of dimension studied (Hausdorff, Packing, Assouad, L^q, ...)

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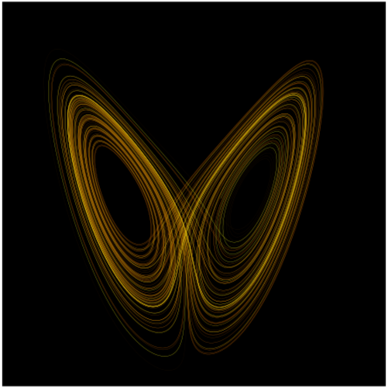
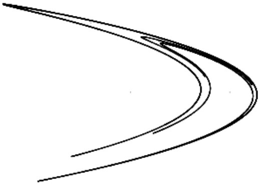
- How to distinguish sets with dimensions?
- What (metric, topological, etc.) properties do dimensions influence, or influence dimensions?
- Connections to harmonic analysis, etc. (Projections of sets, Kakeya conjecture, ...)

Dynamical Systems





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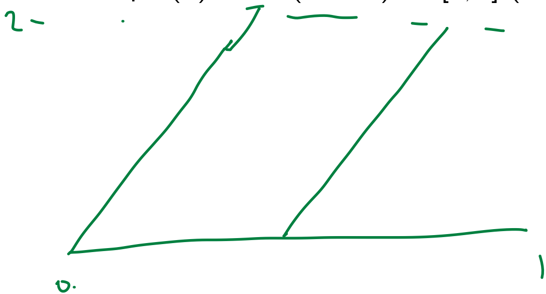
- Interested in: long term behaviour of the system from initial condition
  - describe trajectory based on initial conditions
  - where does the trajectory end up / spend most time?

Doubling Map

- Consider map $f(x) = 2x \pmod 1$ on $[0, 1]$ (“doubling map”)



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- can study sequences $\{0,1\}^{\mathbb{N}}$ (as encoding points in $[0,1]$)

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$$x = \frac{p}{q} \qquad 2x = \frac{2p \pmod q}{q}$$

$$2^k x = \frac{2^k p \pmod q}{q}$$

Orbits

$x \quad 2x \quad 2^2x \quad 2^3x \quad \ldots$

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  - Point 0.100000010000000000000000000000000000000001... does not have dense orbit

Ergodic Theorems: typical behaviour

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$$\{x, f(x), f \circ f(x), \ldots, f^{(n-1)}(x)\}$$

are in $[a, b]$.

$x \quad 2x \quad 2^2 x \quad \cdot \quad 2^{n-1} x$

$b - a$

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- with probability 1, a random point has dense orbit
- But there are many points without dense orbits!

Further directions

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- Number theory
  - Doubling map: properties of decimal expansions
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- important tool in many other areas of maths / analysis