Multifractal Analysis and the Geometry of Lagrange Multipliers

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St Andrews Research Day.

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Given a dynamical system or PDE defining system evolution...

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Given a dynamical system or PDE defining system evolution...

Given a probability distribution or measure ...

Common perspective

What can one say about typical properties?

iid sequence $\Longrightarrow$ SLLN

dynamics $\Longrightarrow$ ergodic theorems

measure $\Longrightarrow$ dimension

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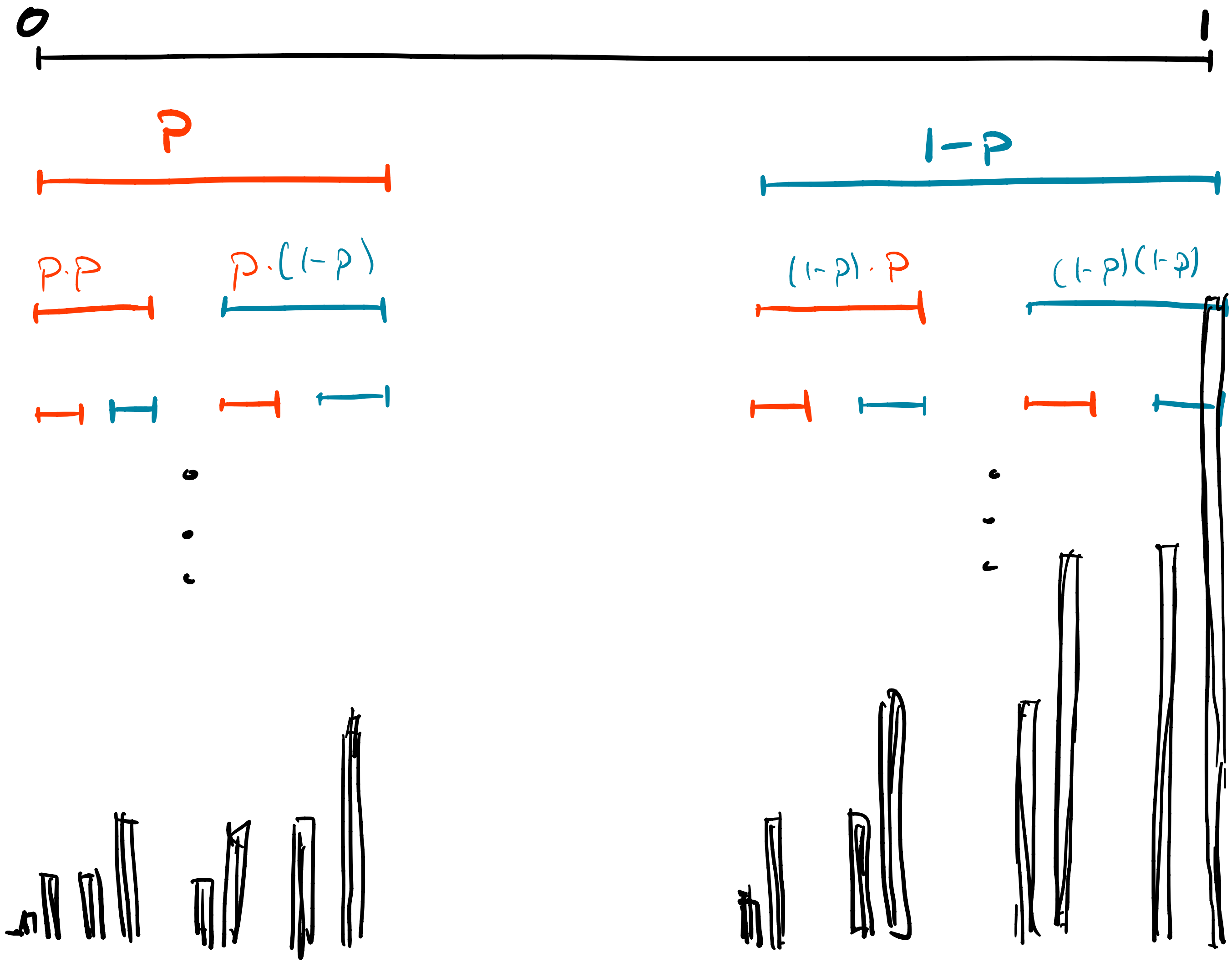
* What about non-typical properties? *

$\Longrightarrow$ large deviations or multifractal analysis.



0
1
p
1-p
p·p
p·(1-p)
(1-p)·p
(1-p)(1-p)

Multifractal analysis of measures:

Consider measure ν on $[0,1]$ ("fractal" measure)

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What about other values of $\alpha \neq \dim(\nu)$?

$$f(\alpha) = \dim\{x \in K : \dim(\mu, x) = \alpha\}$$

$$\dim(\mu, x) = \lim_{r \to 0} \frac{\log \mu B(x,r)}{\log r}$$

$$\geqslant \max_{\nu \in \Delta} \{\dim(\nu) : \mathrm{rel}(\nu, \mu) = \alpha\}$$

ν-typical values of $\dim(\mu, x)$

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α

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CONSTRAINED OPTIMIZATION PROBLEM

Δ — domain (space of measures)

$\dim(\;)$ — objective function

$\mathrm{rel}(\;) = \alpha$ — constraint

$$\max_{\nu \in \Delta} \{ \dim(\nu) : \mathrm{rel}(\nu) = \alpha \} = F(\alpha)$$

HAS DUAL

$$\min_{\nu \in \Delta} \{ q \cdot \mathrm{rel}(\nu) - \dim(\nu) \} = T(q)$$

Lemma: $F(\alpha) \leq T^*(\alpha)$ (assume Δ compact; dim upper semict)

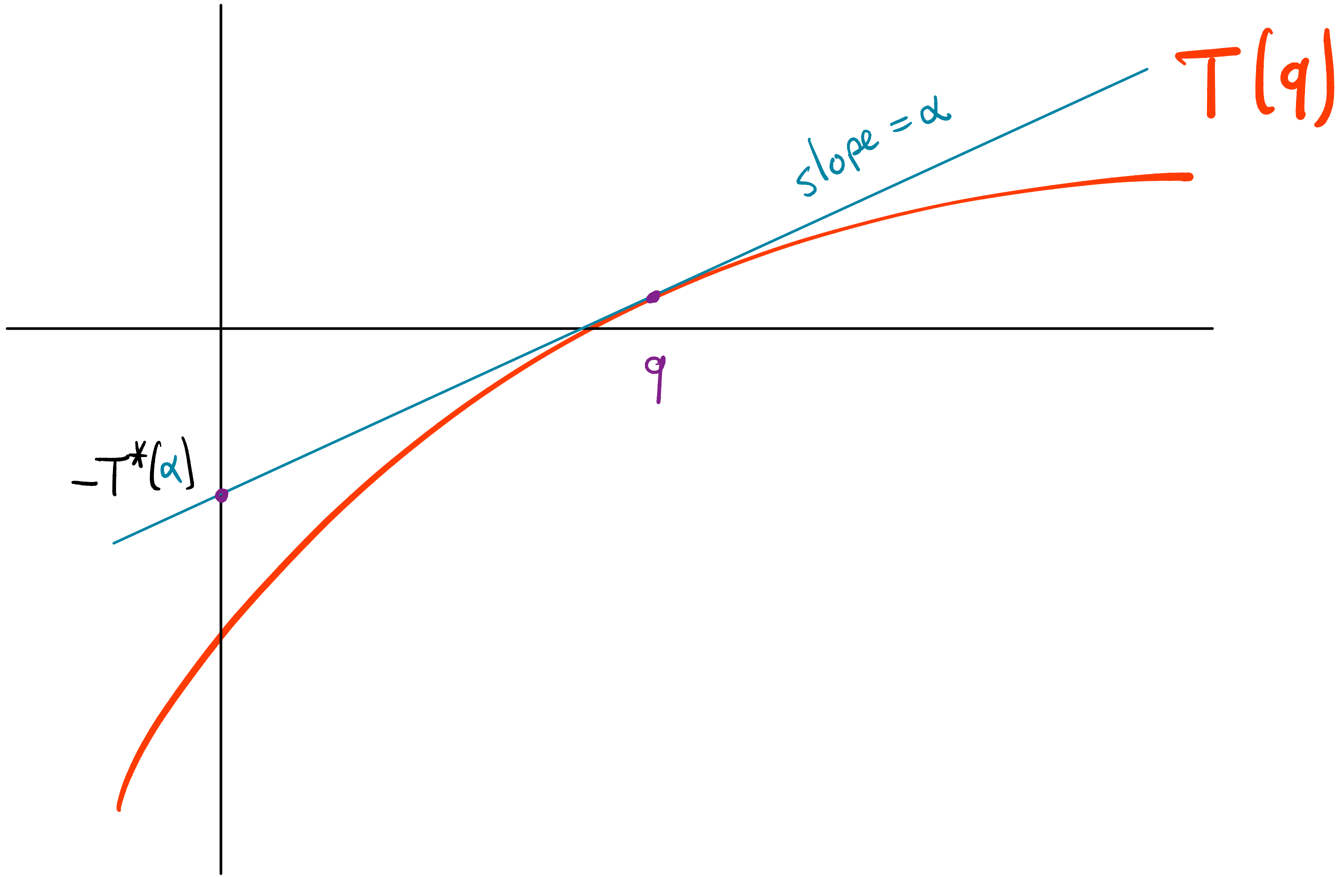
Let $\nu \in \Delta$ s.t.

$$F(\alpha) = \dim(\nu)$$
$$\alpha = \mathrm{rel}(\nu)$$

$$\Longrightarrow T(q) \leq q\,\mathrm{rel}(\nu) - \dim(\nu)$$
$$= q\alpha - F(\alpha)$$

(q Arbitrary)

$$\Longrightarrow F(\alpha) \leq \inf_{q \in \mathbb{R}} \left(q\alpha - T(q) \right) \stackrel{\text{def}}{=} T^*(\alpha)$$



$T(q)$
slope $= \alpha$
q
$-T^*(\alpha)$

$$F(\alpha) \leq T^*(\alpha)$$

OTHER DIRECTION?

Problem: need to choose minimizer v for $T(q)$ s.t. $\mathrm{rel}(v) = \alpha$. Not possible in general!

[Note: $F(\alpha) = T^*(\alpha)$ if $T'(q) = \alpha$ exists]

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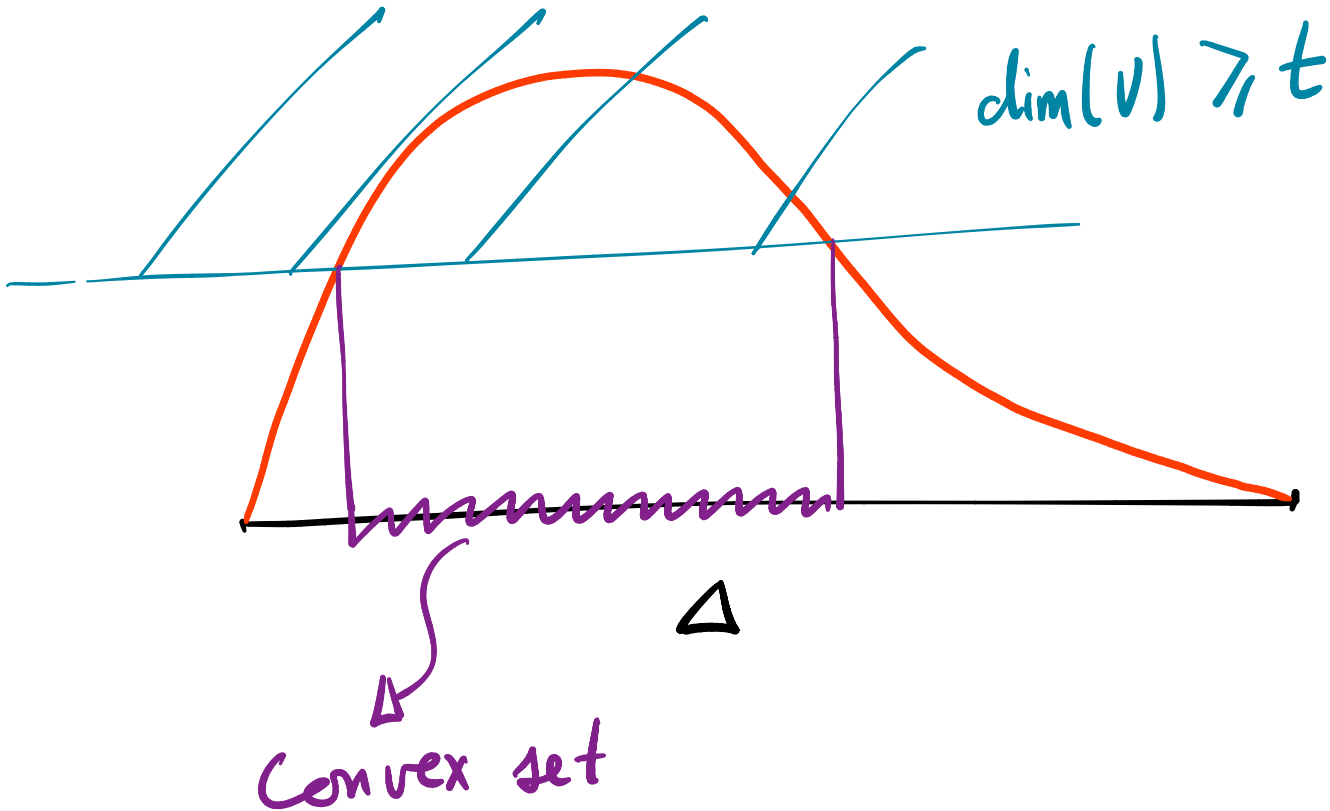
$$\left\{ \nu : \frac{h(\nu)}{\lambda(\nu)} \geqslant t \right\} = \{ \nu : \underbrace{h(\nu) - t \cdot \lambda(\nu)}_{\text{strictly concave}} \geqslant 0 \}$$

is a CONVEX SET


dim(V) ⩾ t
Δ
convex set

QUASICONCAVE OBJECTIVE

$$\Downarrow \text{ (in general)}$$

$T(q)$ UNIQUE MINIMIZER

$$\Downarrow \text{ (in general)}$$

$$F(\alpha) = T^*(\alpha)$$

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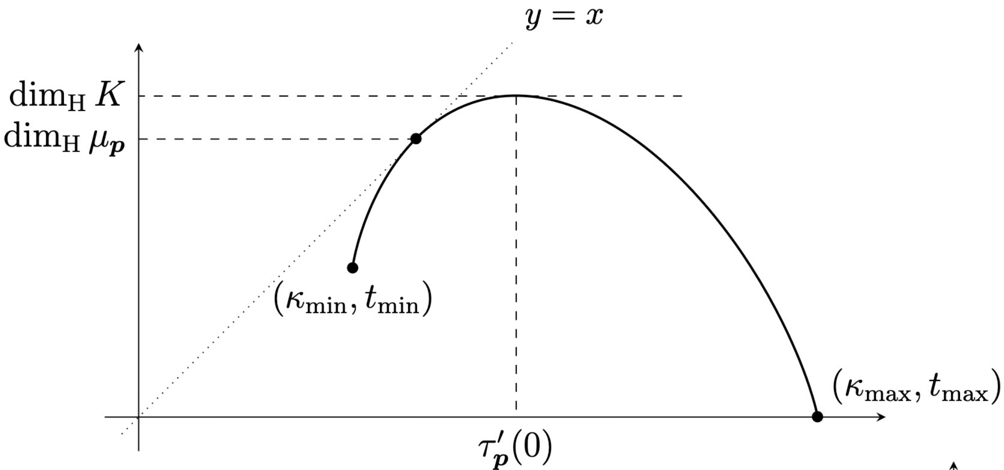
special

"general" bounds

$$T^*(\alpha) = F(\alpha) \leqslant f(\alpha) \leqslant T^*(\alpha)$$

ALL EQUALITIES!



$= f(\alpha) = F(\alpha)$

$\tau(q) = T(q) \quad =$