Continued fractions and self-conformal sets

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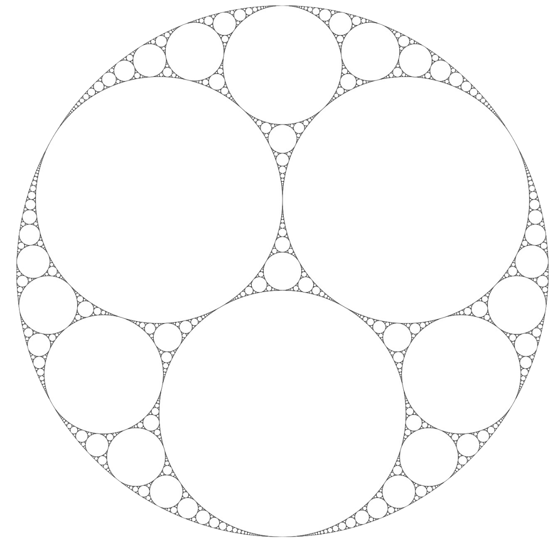
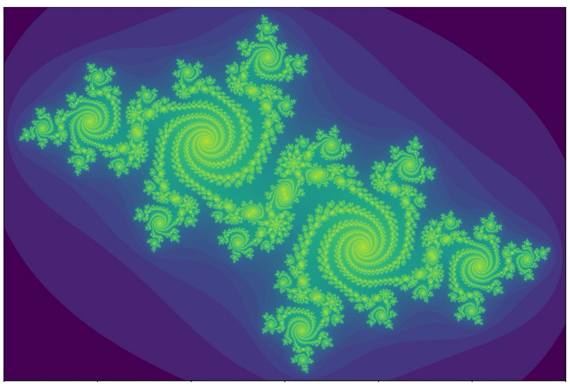
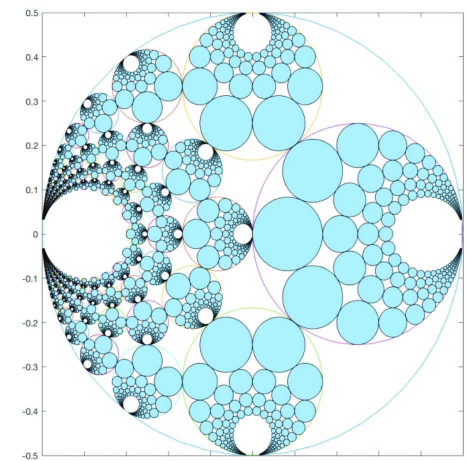
Fractal Geometry?

Study of sets with (a priori)
no smooth structure







- existence of intrinsic structures
  - → e.g. Haar measure
- invariants for families of maps
  - → e.g. dimension (of manifold)
- regularity
  - → e.g. smoothness or continuity

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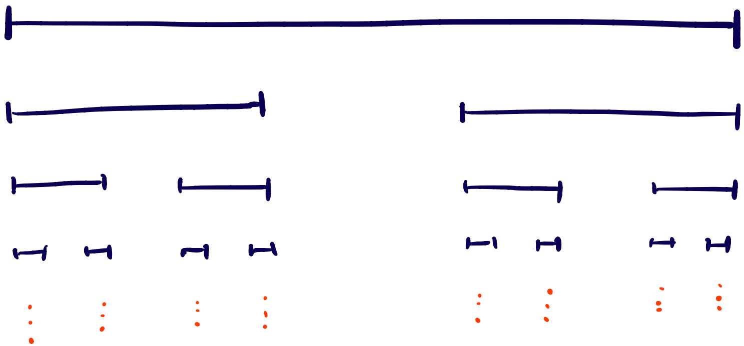
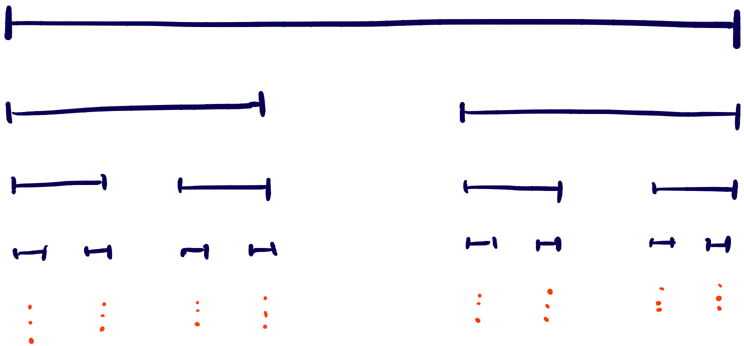
A classical fractal set



(middle thirds) Cantor set

A classical fractal set



(middle thirds) Cantor set

$$C = \left\{ x \in [0,1] : \begin{array}{l} \bullet\ x = 0.a_1 a_2 a_3 \ldots \text{ in base 3} \\ \bullet\ a_i \in \{0,2\} \end{array} \right\}$$

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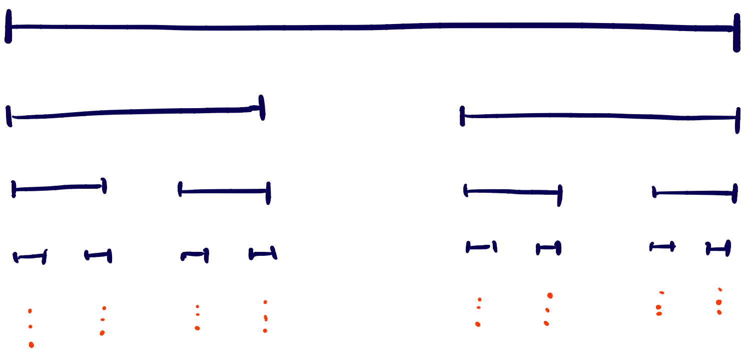


(middle thirds) Cantor set

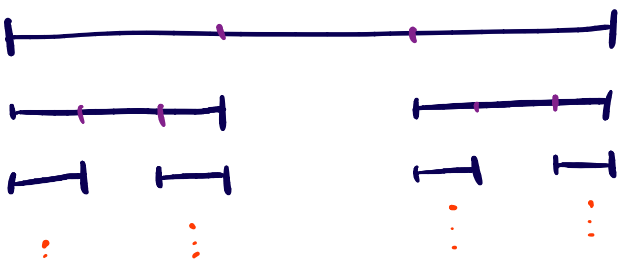
$C = \{x \in [0,1]:$

- $b \geq 2$
- $x = 0.a_1 a_2 a_3 \ldots$ in base b
- $a_i \in A \}$

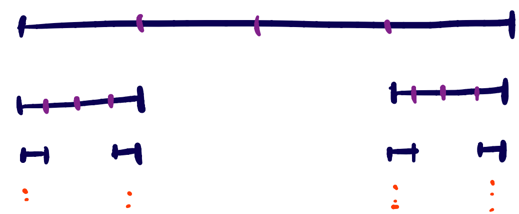
missing digit set

How to distinguish between Cantor sets?

- $b = 3$
- $A = \{0, 2\}$



- $b = 4$
- $A = \{0, 3\}$



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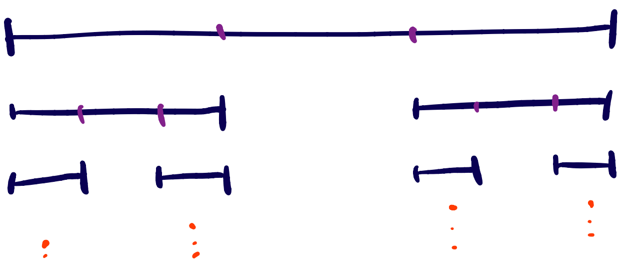

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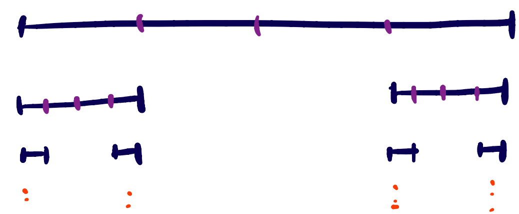

- homeomorphic
- not bi-Lipschitz equivalent

How to distinguish? Fractal dimension(s)

- $r \in (0,1)$ $\quad N_r(E) = \left(\text{least \# of balls of radius } r \text{ required to cover } E\right)$

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- $$\overline{\dim}_B E = \limsup_{r \to 0} \frac{\log N_r(E)}{\log(1/r)}$$

$$N_{3^{-n}}(C) \approx 2^n$$

$$\frac{\log N_{3^{-n}}(C)}{\log(1/3^{-n})} \to \frac{\log 2}{\log 3}$$

- $\underline{\dim}_B E = \liminf\limits_{r \to 0} \dfrac{\underline{\log} N_r(E)}{\log(1/r)}$

- $\dim_B E$ is bi-Lipschitz invariant

$$\left.\begin{aligned} \dim_B C_3 &= \frac{\log 2}{\log 3} \\ &\ne \\ \dim_B C_4 &= \frac{\log 2}{\log 4} \end{aligned}\right\} \Rightarrow C_3 \text{ and } C_4$$

are not bi-Lipschitz equivalent

$$\frac{\log N_r(E)}{\log(1/r)} = s \implies \exists \ x_1, \dots, x_{N_r(E)} \text{ s.t.}$$

$$N_r(E) = \left(\frac{1}{r}\right)^s$$

(1) $E \subset \bigcup_{i=1}^{N_r(E)} B(x_i, r)$

(2) $\sum_{i=1}^{N_r(E)} r^s = 1$ ⟶ "s-cost"

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(2) $\sum_{i=1}^{N_r(E)} r^s = 1 \rightsquigarrow$ "s-cost"

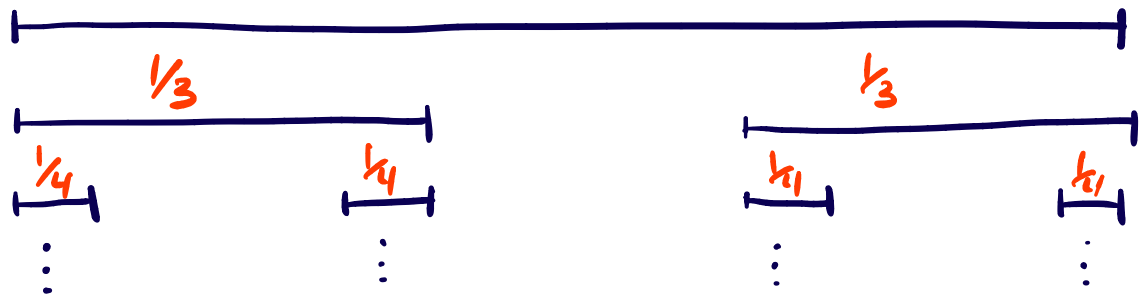
Hausdorff dimension: allow arbitrary covers

$$\dim_H E = \inf \Big\{ s : \forall \delta > 0 \; \exists \; \{B(x_i, r_i)\} \text{ s.t.}$$

(1) $E \subset \bigcup_i B(x_i, r_i)$

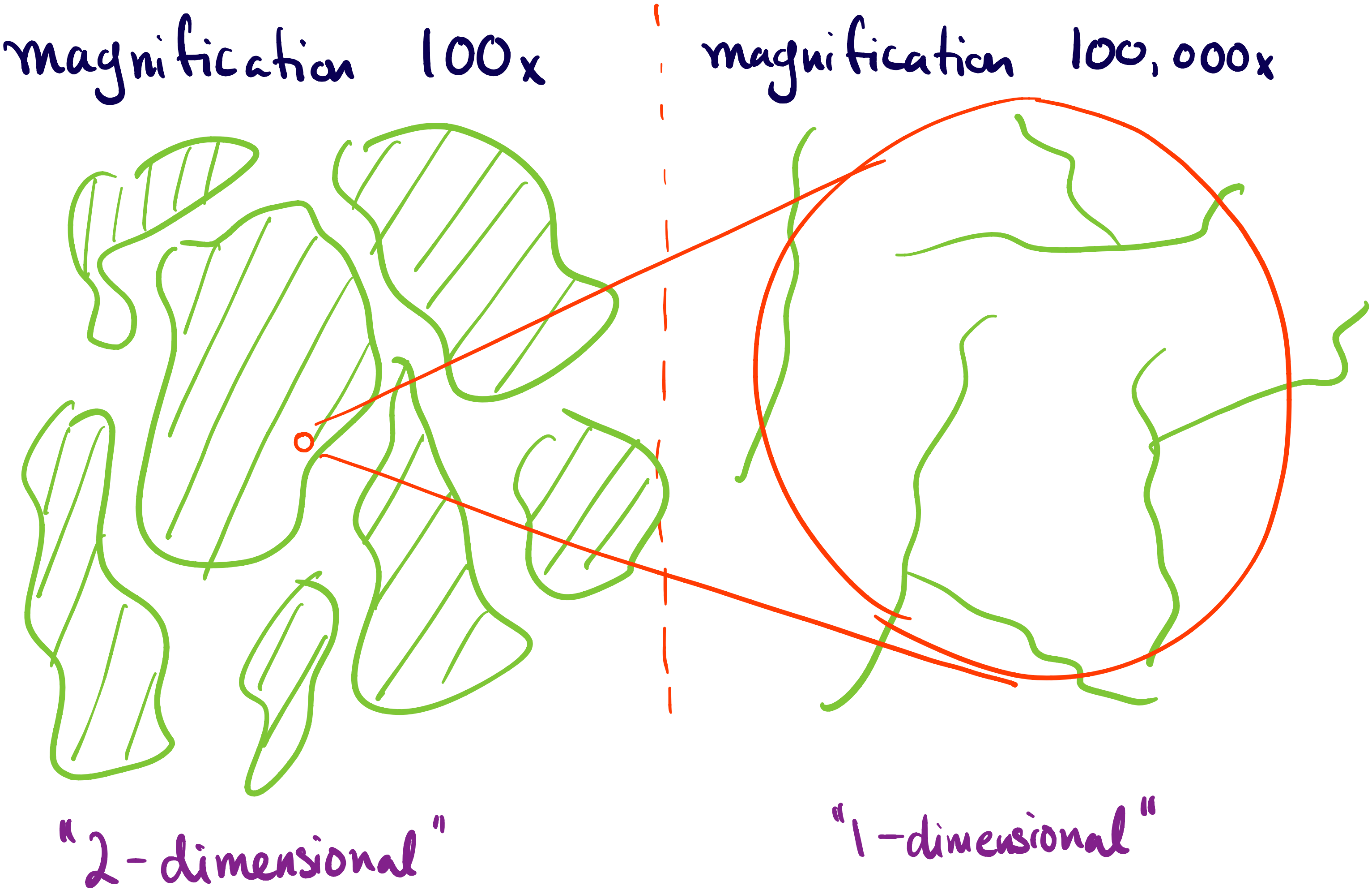
(2) $\sum r_i^s \leqslant 1 \Big\}$

- limit does not need to exist



$$\underbrace{\frac{1}{3},\ \dots,\ \frac{1}{3}}_{N_1},\ \underbrace{\frac{1}{4},\ \dots,\ \frac{1}{4}}_{N_2},\ \underbrace{\frac{1}{3},\ \dots,\ \frac{1}{3}}_{N_3}$$

$$N_1 \ll N_2 \ll N_3 \ll \cdots$$

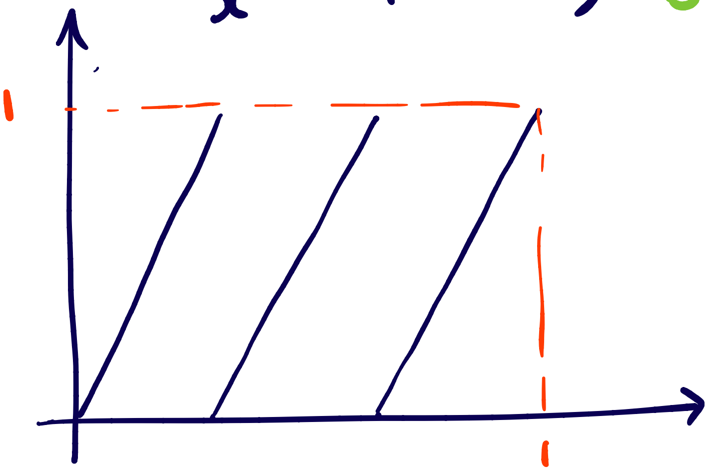

magnification 100x
magnification 100,000x
"2-dimensional"
"1-dimensional"

In general, $\dim_H E \leqslant \underline{\dim}_B E \leqslant \overline{\dim}_B E$

When do dimensions coincide?

$$f: [0,1] \longrightarrow [0,1]$$

$$x \longmapsto b \cdot x \pmod 1$$



$S^1 = \mathbb{R}/\mathbb{Z}$

$$f: S^1 \longrightarrow S^1$$

Smooth

$f(x) = b \cdot x \pmod 1$

Recall:
- $b \geq 2$ base
- $A \subset \{0, \dots, b-1\}$ allowed digits
- $C_{b,A} = \{x = 0.a_1 a_2 a_3 : a_i \in A\}$

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- $b \geq 2$ base
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- $C_{b,A} = \{x = 0.a_1 a_2 a_3 : a_i \in A\}$

$$f(0.a_1 a_2 a_3 \dots) = 0.a_2 a_3 a_4 \dots$$

$\Rightarrow f(C_{b,A}) = C_{b,A} \rightsquigarrow C_{b,A}$ invariant for f.

Abstract setup:

- M Riemannian manifold; $f: M \longrightarrow M$
- f conformal: $Df(x) = \text{constant} \cdot \text{orthogonal matrix}$
- f uniformly expanding: $\exists\, C > 0,\ \lambda > 1$

  $\| Df^n(x) \| \geq C \cdot \lambda^n$

Examples:

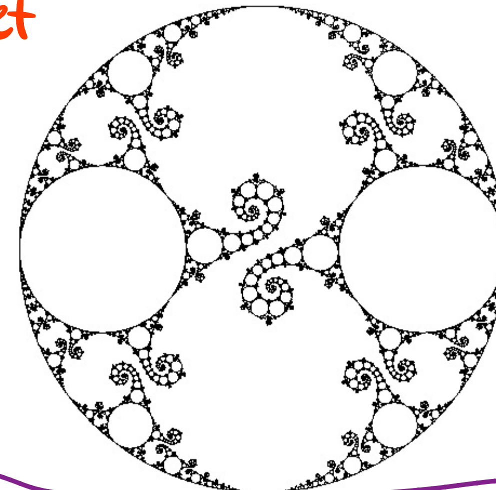
- Newton iteration: for polynomials $\quad x \longmapsto x - \dfrac{p(x)}{p'(x)}$

- Accumulation pts of action of Kleinian group (discrete subgroup of orientation-preserving isometries of hyperbolic space).

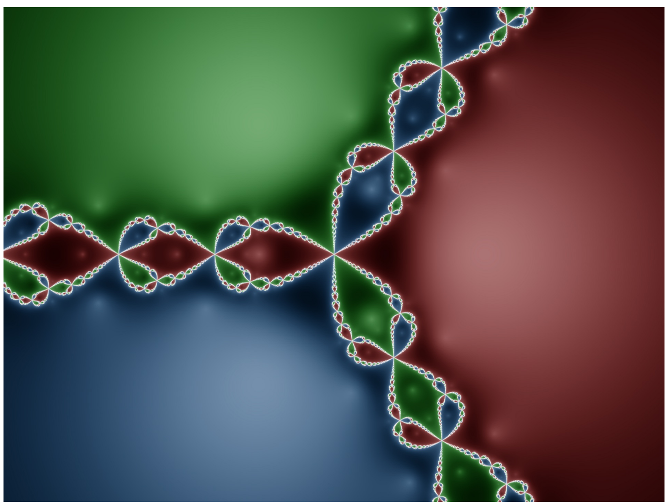
- Self-conformal sets: $\quad K = \bigcup_{i=1}^{m} \varphi_i(K)$

  φ_i conformal

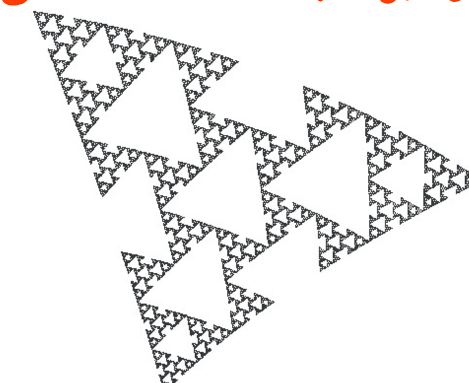
limit set of Kleinian gp.



Basins of attraction for Newton iteration of x^3



self-conformal set



Theorem (Falconer '89; Barreira '96; Gatzouras – Peres '97)

Suppose

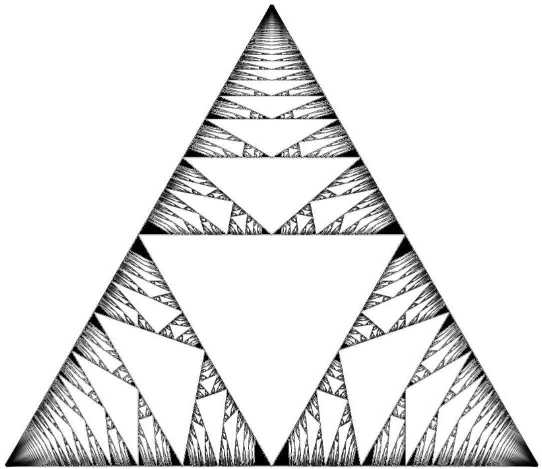
- $f: M \to M$ conformal
- $\Lambda \subset M$ compact + invariant $[f(\Lambda) = \Lambda]$
- f uniformly expanding on Λ

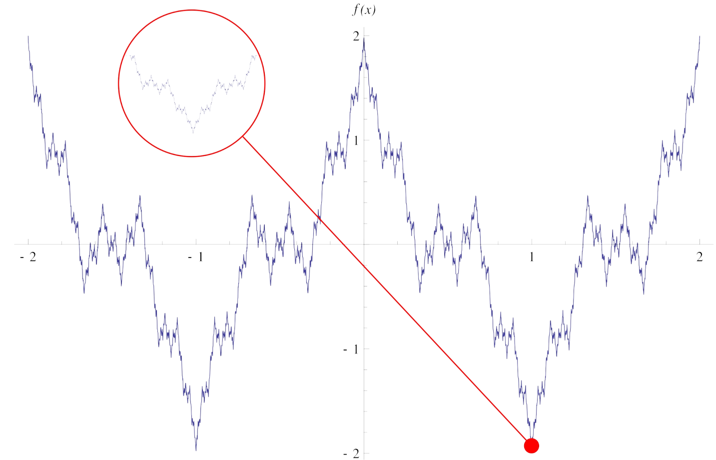
Then $\dim_H \Lambda = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$

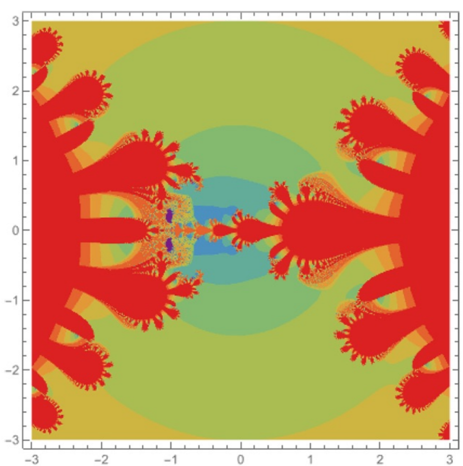
"has the same size at all scales"

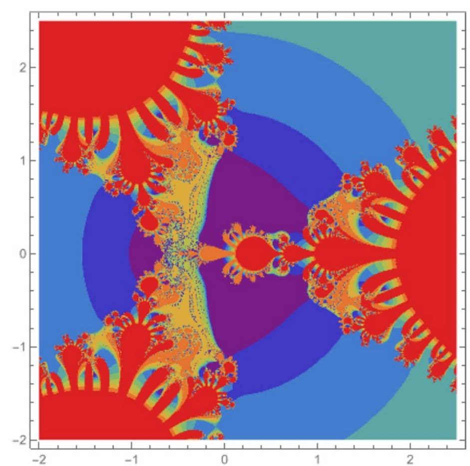
Which assumptions are required?

- non-conformal:
  - $\dim_H \Lambda < \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$
    - Bedford '84
    - McMullen '84
    - Stallard '01 '04
  - $\underline{\dim}_B \Lambda < \overline{\dim}_B \Lambda$
    - Jurga '23
- non-uniformly expanding:
  - $\dim_H \Lambda < \dim_B \Lambda$
    - Mauldin-Urbański '99




f(x)
x
- 2
- 1
1
2
2
1
- 1
- 2


3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3


2
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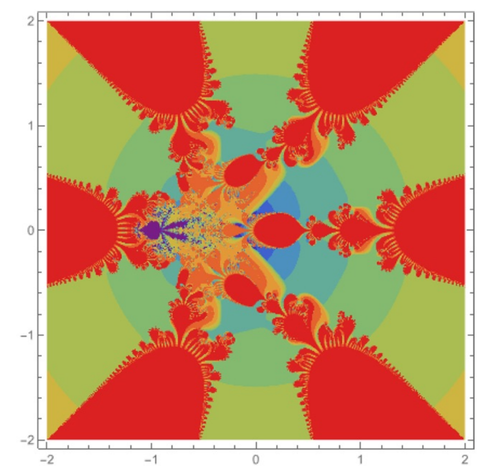
−1
0
1
2

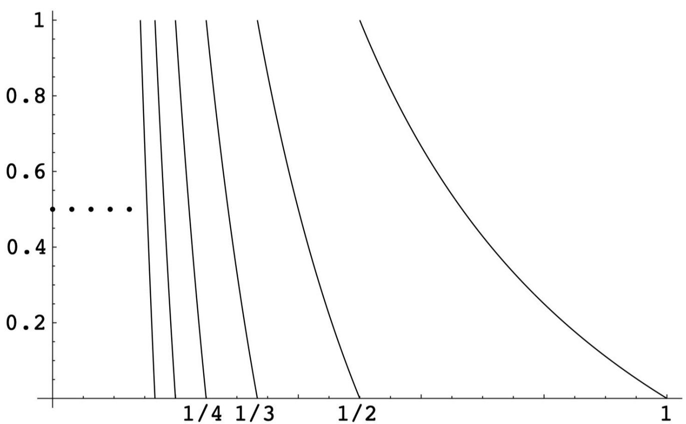

2
1
0
−1
−2
−2
−1
0
1
2

What about Compactness?

Continued fraction expansion

$$x = [a_1, a_2, a_3, \ldots] \quad \sim \quad x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}}$$

$$\varphi(x) = \frac{1}{x} \pmod 1$$



$\varphi([a_1, a_2, a_3, \dots])$
$= [a_2, a_3, a_4, \dots]$

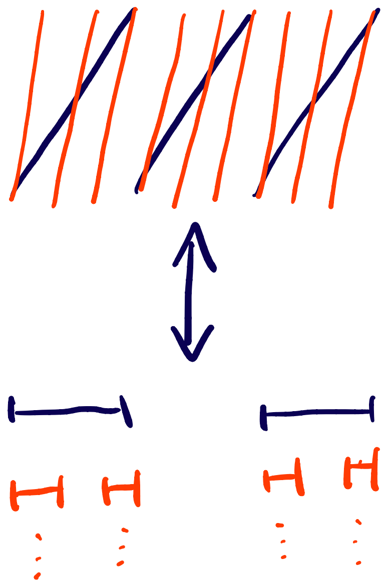
$x = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \ddots}}} \sim x = [a_1, a_2, a_3, \dots]$

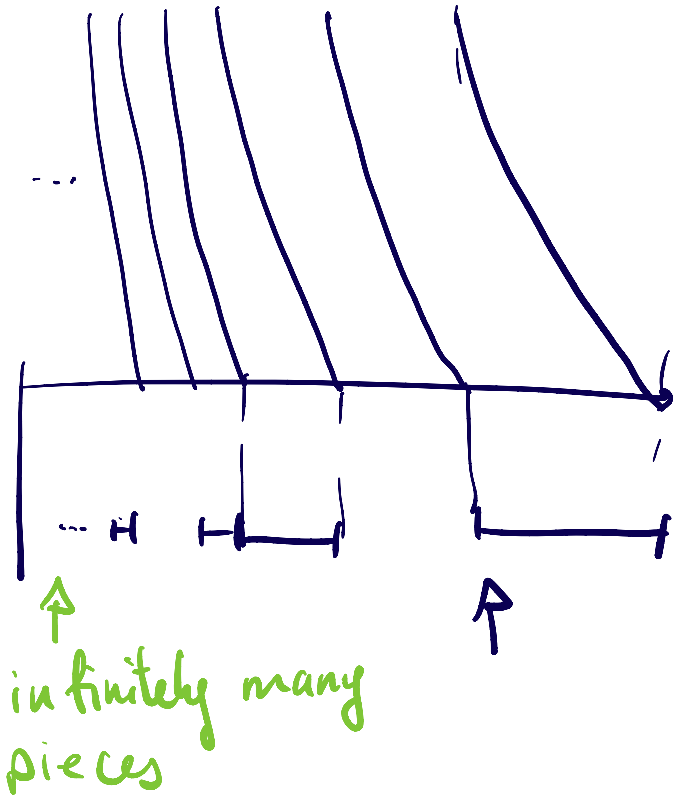
Suppose $A \subset \mathbb{N}$:

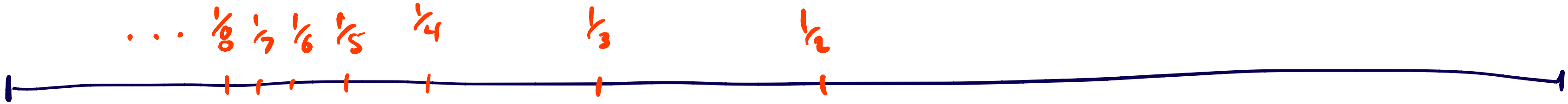
$$\Lambda_A = \{x = [a_1, a_2, a_3, \dots] : a_i \in A\}$$

Then:
- $\varphi(\Lambda_A) = \Lambda_A$
- φ conformal + uniformly expanding on Λ_A
- HOWEVER Λ_A NOT CLOSED (unless A finite)

We can "invert" the branches and study the set by subdivision.






. . .
1/8
1/7
1/6
1/5
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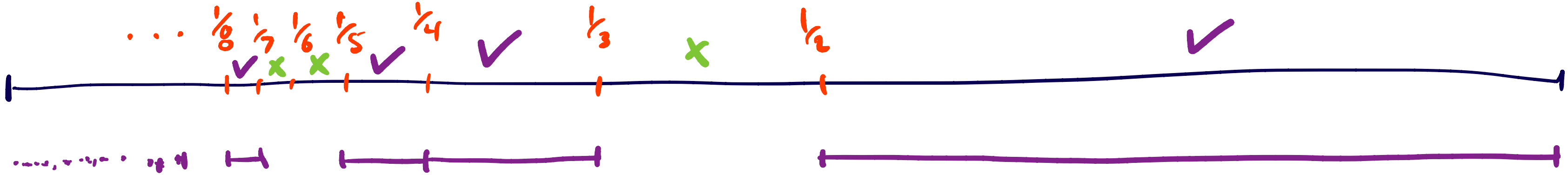
1/2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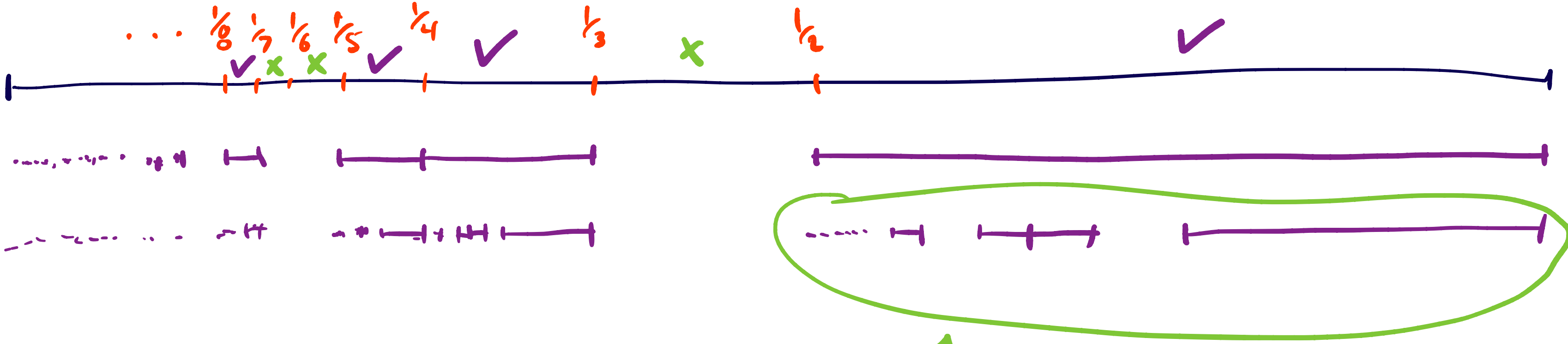
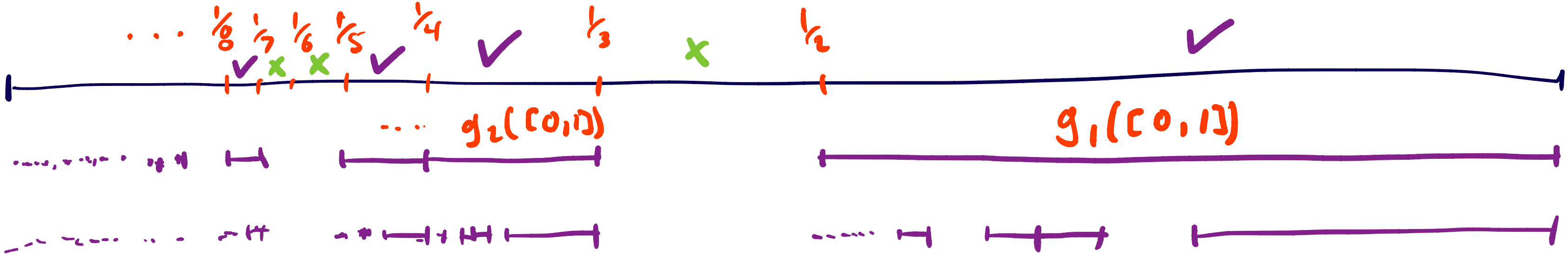


image of Λ_A (up to some small distortion)



What are dimensions of K?

Natural cover: use <u>intervals</u> from construction

$$K_1 = \bigcup_{a \in A} g_a([0,1])$$

$$K_2 = \bigcup_{a \in A} \bigcup_{b \in A} g_a \circ g_b([0,1])$$

$$\vdots$$

$\dim_H K \leqslant h$ where $\underbrace{\sum_{a \in A} |g_a([0,1])|^h \leqslant 1}$

same formula as in Cantor set case

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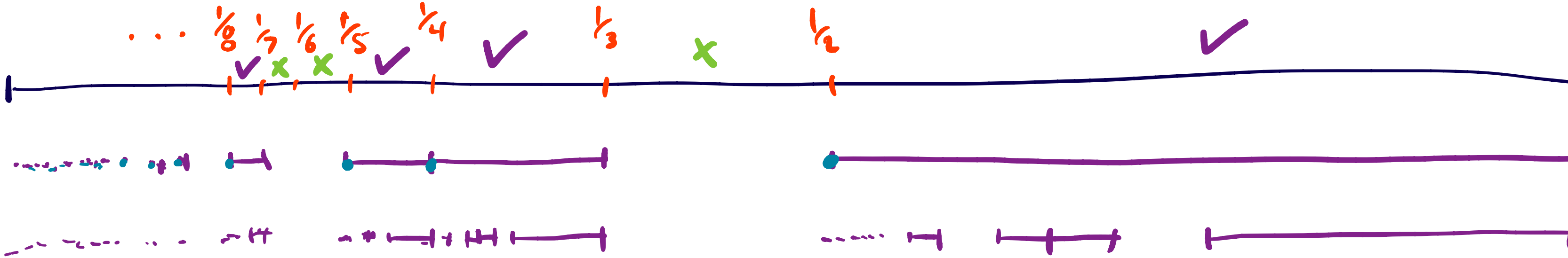
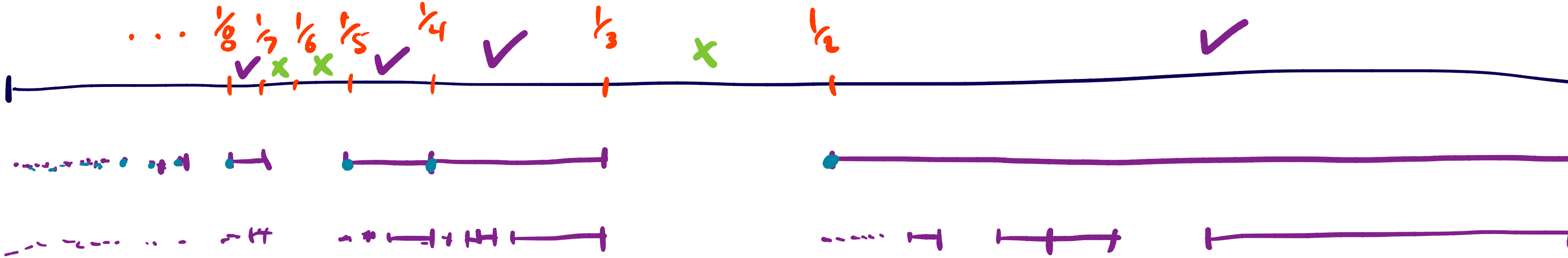
BUT: the intervals $g_a([0,1])$ have width $\longrightarrow 0$, and there are infinitely many of them!



Another obstruction: accumulation rate at 0

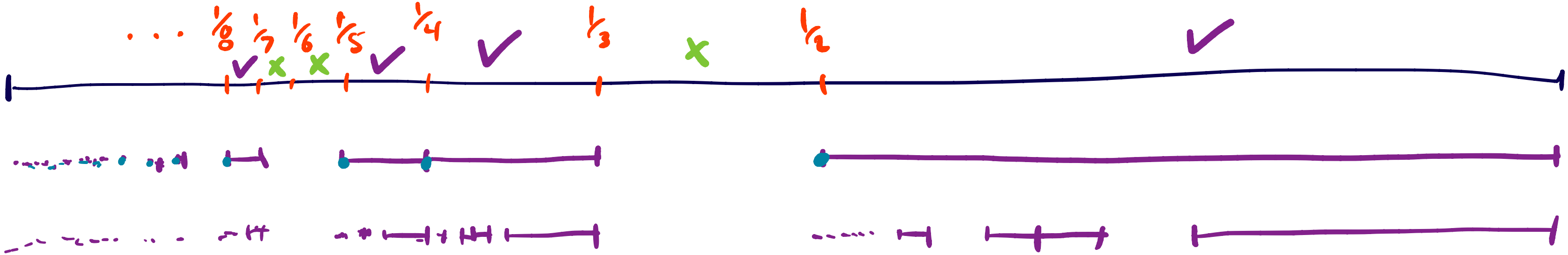


Another obstruction: accumulation rate at 0

Let $F = \{ g_a(0) : a \in A \}$ = left endpoints.

Then $F \subset \Lambda$ so

- $\overline{\dim}_B \Lambda \geq \overline{\dim}_B F$
- $\underline{\dim}_B \Lambda \geq \underline{\dim}_B F$



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$F \subset \{ \frac{1}{a} : a \in \mathbb{N} \} \Rightarrow 0 \leq \overline{\dim}_B F \leq 1/2$

Theorem (Mauldin–Urbański)

'96 $\quad \dim_H \Lambda = h$

'99 $\quad \overline{\dim}_B \Lambda = \max\{h, \overline{\dim}_B F\}$

In particular, $\dim_H \Lambda < \overline{\dim}_B \Lambda$ is possible.

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'99 $\overline{\dim}_B \Lambda = \max\{h, \overline{\dim}_B F\}$

In particular, $\dim_H \Lambda < \overline{\dim}_B \Lambda$ is possible.

What about $\underline{\dim}_B \Lambda$?

Questions:

(1) Does $\underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$?

(2) If not, what can be said about $\underline{\dim}_B \Lambda$?

(2') Does $\underline{\dim}_B \Lambda$ depend only on

$$\{\dim_H \Lambda,\ \underline{\dim}_B F,\ \overline{\dim}_B F\}?$$

Recall for general sets E:

$$\dim_H E \leqslant \underline{\dim}_B E \leqslant \overline{\dim}_B E$$

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$$\dim_H E \leqslant \underline{\dim}_B E \leqslant \overline{\dim}_B E$$

Apply Mauldin–Urbański:

$$\max\{\dim_H \Lambda, \underline{\dim}_B F\}$$
$$\leqslant \underline{\dim}_B \Lambda$$
$$\leqslant \overline{\dim}_B \Lambda$$
$$= \max\{\dim_H \Lambda, \overline{\dim}_B F\}$$

Recall for general sets E:

$$\dim_H E \leqslant \underline{\dim}_B E \leqslant \overline{\dim}_B E$$

Apply Mauldin–Urbański '99:

$$\left.\begin{aligned} &\max\{\dim_H \Lambda, \underline{\dim}_B F\} \\ &\leqslant \underline{\dim}_B \Lambda \\ &\leqslant \overline{\dim}_B \Lambda \\ &= \max\{\dim_H \Lambda, \overline{\dim}_B F\} \end{aligned}\right\} \Rightarrow$$

If

- $\underline{\dim}_B F = \overline{\dim}_B F$
- $\dim_H \Lambda \geqslant \overline{\dim}_B F$

then

$$\underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$$

Theorem (Banaji–R., 2024+)

$$\underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$$

iff

$$\underline{\dim}_B F = \overline{\dim}_B F \quad \text{OR} \quad \dim_H \Lambda \geq \overline{\dim}_B F$$

Theorem (Banaji–R., 2024+)

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iff

$$\underline{\dim}_B F = \overline{\dim}_B F \quad \text{OR} \quad \dim_H \Lambda \geq \overline{\dim}_B F$$

In particular, it can happen that

$$\underline{\dim}_B \Lambda < \overline{\dim}_B \Lambda$$

Theorem (con't.)

Sharp bounds: (if $h < \overline{\dim}_B F$; otherwise $h = \underline{\dim}_B \Lambda = \overline{\dim}_B \Lambda$)

$$\underbrace{\max\{h, \underline{\dim}_B F\}}_{\text{trivial lower bound}} \leq \underline{\dim}_B \Lambda \leq h + \underbrace{\frac{(\overline{\dim}_B F - h)(1-h)\cdot \underline{\dim}_B F}{\cdot\overline{\dim}_B F - h\cdot \underline{\dim}_B F}}_{\text{only depends on } \overline{\dim}_B F,\ h,\ d,\ \underline{\dim}_B F}$$

Theorem (con't.)

Sharp bounds:

$$\underbrace{\max\{h, \underline{\dim}_B F\}}_{\text{trivial lower bound}} \leqslant \underline{\dim}_B \Lambda \leqslant h + \underbrace{\frac{(\overline{\dim}_B F - h)(d-h)\cdot \underline{\dim}_B F}{d \cdot \overline{\dim}_B F - h \cdot \underline{\dim}_B F}}_{\text{only depends on } \overline{\dim}_B F,\ h,\ d,\ \underline{\dim}_B F}$$

Moreover: Any configuration satisfying this inequality is permitted (i.e. $\underline{\dim}_B \Lambda$ is not a fn of $\overline{\dim}_B F$, h, d, $\underline{\dim}_B F$)

Non-compactness is <u>essential</u>

(still conformal + uniformly expanding)

Thank you!