Exercise 1

DUE 12:15PM ON THURSDAY, JANUARY 15

Important information.

- Please submit the exercise sheets by email to `alex@rutar.org` as a PDF. LaTeX'd (or otherwise typeset) solutions are greatly appreciated; if they are scans please make sure that the scanned pages are sorted and the writing is easily legible.
- The deadline is strict since I will grade the sheets before the exercise classes, which take place on Fridays after the lecture. If you have extenuating circumstances, let me know and we will work something out.
- Working together is encouraged! However, please submit your own solutions and clearly indicate any collaboration on the assignment sheet. Everyone will be required to present something during the exercise classes, so be prepared to explain anything that you have written. I will not ask you to present anything which you did not successfully complete in your submitted solutions.
- Each exercise sheet will be graded out of 10 points, with 1 additional bonus point. Your grade will depend primarily on your written response, but also on your presentation during the exercise class. The presentation is intended in part as preparation for the final project presentation. Excess bonus points will count as bonus points towards the final project!
- The number of available points for a given question does not necessarily reflect the difficulty of the question.
- Feel free to email me at `alex@rutar.org` with questions or for hints if you are stuck! If I give hints to anybody, I will share them with everyone.

The questions.

1. *(3 pt.)* Let (X, d) be a complete metric space and let $\Phi = \{f_i\}_{i\in\mathcal{I}}$ be an IFS. Fix some $z \in X$.
   (i) Prove that there exists a number $R \geq 0$ so that $f_i(B(z, R)) \subset B(z, R)$ for all $i \in \mathcal{I}$.
   (ii) Let Q be any non-empty compact set such that $f_i(Q) \subset Q$ for all $i \in \mathcal{I}$. For each $n \geq 0$, define

$$K_n = \bigcup_{\mathtt{i}\in\mathcal{I}^n} f_{\mathtt{i}}(Q).$$

Prove that $K_0 \supset K_1 \supset K_2 \supset \cdots$ is a nested sequence of compact sets.

(iii) Prove that $K = \bigcap_{n=1}^{\infty} K_n$ is the attractor of the IFS Φ.

(iv) Let $X = \mathbb{R}^2$ and consider the IFS defined by the three maps $f_1(x,y) = (x/2, y/2)$, $f_2(x,y) = (x/2 + 1/3, y/2 + 1/4)$, $f_3(x,y) = (x/2 + 1/2, y/2 + 1/2)$. Let $Q = [0,1]^2$. Observe that $f_i(Q) \subset Q$ for each $i = 1,2,3$. Sketch (by hand, or using a computer) a few of the sets K_n (say, up to K_3 by hand or K_8 with a computer) using the initial compact set Q.

2. *(5 pt.)* Let (X,d) be a complete metric space, let $\Phi = \{f_i\}_{i\in\mathcal{I}}$ be an IFS, and let F be an arbitrary compact set.

(i) Prove that there is a unique non-empty compact set K_F such that

$$K_F = \bigcup_{i\in\mathcal{I}} f_i(K_F) \cup F.$$

(ii) For each integer $n \geq 0$, define

$$F^{(n)} = \bigcup_{k=0}^{n} \bigcup_{\mathtt{i}\in\mathcal{I}^k} f_{\mathtt{i}}(F).$$

Observe that $F^{(n)} \subset K_F$ for all n (why?). Prove that

$$\lim_{n\to\infty} d_{\mathcal{H}}(F^{(n)}, K_F) = 0.$$

(iii) What is

$$\lim_{k\to\infty} \bigcup_{\mathtt{i}\in\mathcal{I}^k} f_{\mathtt{i}}(F)$$

in the Hausdorff metric? (Just state the answer, you don't need to include all of the details of the proof.)

(iv) Prove that $K_F = K_{\varnothing}$ if and only if $F \subset K_{\varnothing}$.

(v) Take the IFS from Q1(iv). Let $F = \{(0,0)\}$ be the set consisting of the unique fixed point of the map f_1. Sketch (by hand, or using a computer) a few of the sets $F^{(n)}$ (say, up to $F^{(4)}$ by hand or $F^{(8)}$ with a computer).

3. *(2 pt.)* Let $f_1(x) = x/3$ and $f_2(x) = x/3 + 2/3$ be the IFS generating the middle-thirds Cantor set C. Call a set $E \subset \mathbb{R}$ *invariant* if E is non-empty and $E = f_1(E) \cup f_2(E)$. Recall that C is the unique compact invariant set.

(i) Give an example of a *non-compact* and *bounded* invariant set E.

(ii) Give an example of a *non-compact* and *closed* invariant set E.

(iii) Explain why every invariant set E must contain infinitely many points.

4. **(1 pt. bonus)** Let (X,d) be a non-empty *compact* metric space and suppose $f\colon X \to X$ satisfies $d(f(x), f(y)) < d(x,y)$ for all $x \neq y$ in X. Prove that there is a unique $x_* \in X$ such that $f(x_*) = x_*$, and moreover $x_* = \lim_{n\to\infty} f^n(x)$ for any starting point $x \in X$. *Hint: choose $x_* \in X$ to minimize $x \mapsto d(x, f(x))$.*

Aside: using this, one can prove that there is a unique attractor even if one replaces uniform contraction with contraction in the definition of an IFS.