Exercise 3

DUE 12:15PM ON THURSDAY, JANUARY 29

The questions.

1. *(3 pt.)* Let (X, d) be a compact metric space.
   (i) Fix an IFS $\{f_i\}_{i\in\mathcal{I}}$ and a probability vector $\boldsymbol{p} \in \mathcal{P}(\mathcal{I})$. Let $z \in X$ be fixed. Using the contraction mapping principle, prove that
   $$\mu_{\boldsymbol{p}} = \lim_{n\to\infty} \sum_{\mathtt{i}\in\mathcal{I}^n} p_{\mathtt{i}} \delta_{f_{\mathtt{i}}(z)}$$
   in the dual Lipschitz metric.
   (ii) Let $F \subset X$ be non-empty and compact. Prove that $\operatorname{diam} F = \operatorname{diam} \mathcal{P}(F)$.
   (iii) Let $Q \subset X$ be a fixed compact set. For each $\mathtt{i} \in \mathcal{I}^*$, let $\nu_{\mathtt{i}}$ be a Borel probability measure with $\operatorname{supp} \nu_{\mathtt{i}} \subset f_{\mathtt{i}}(Q)$. Prove that
   $$\mu_{\boldsymbol{p}} = \lim_{n\to\infty} \sum_{\mathtt{i}\in\mathcal{I}^n} p_{\mathtt{i}} \nu_{\mathtt{i}}$$
   in the dual Lipschitz metric.

2. *(4 pt.)* Let $\{f_i\}_{i\in\mathcal{I}}$ be a self-similar IFS in $\mathbb{R}$ with attractor K. Let $\boldsymbol{p}$ be a probability vector with $p_i > 0$ for all $i \in \mathcal{I}$. Suppose moreover that K is not a singleton. Also, assume that there is a common contraction $r = r_i$ for all $i \in \mathcal{I}$ *(the conclusion is still true without this last assumption, but it simplifies notation a decent amount)*.
   (i) Show that there exists an $m \in \mathbb{N}$ and a $\delta > 0$ such that for all $z \in \mathbb{R}$, there is an $\mathtt{i} \in \mathcal{I}^m$ such that
   $$B(z, \delta) \cap f_{\mathtt{i}}(K) = \varnothing.$$
   (ii) Prove that there is a number $\xi \in (0, 1)$ so that for any $n \in \mathbb{N}$ and $z \in \mathbb{R}$,
   $$\mu_{\boldsymbol{p}}(B(z, \delta r^{(n-1)m})) \le \xi^n.$$
   (iii) Conclude that there is a number $t > 0$ and a constant $C > 0$ so that $\mu_{\boldsymbol{p}}(A) \le C(\operatorname{diam} A)^t$ for all Borel sets A.

3. *(3 pt.)* Let μ be a finite compactly supported Borel measure on $\mathbb{R}^d$ and let $s \in \mathbb{R}$. Prove that $\{x \in \mathbb{R}^d : \underline{\dim}_{\text{loc}}(\mu, x) < s\}$ is a Borel set.

4. **(2 pt. bonus)** Consider the self-similar IFS $\{f_1, f_2, f_3\}$ in $\mathbb{R}$ where $f_1(x) = x/3$, $f_2(x) = x/3 + 2/9$, and $f_3(x) = x/3 + 2/3$. Set $\mathcal{I} = \{1, 2, 3\}$.

(i) Show for all $n \in \mathbb{N} \cup \{0\}$ and $\mathtt{i}, \mathtt{j} \in \mathcal{I}^n$ that either $|f_{\mathtt{i}}(0) - f_{\mathtt{j}}(0)| = 0$ or

$$|f_{\mathtt{i}}(0) - f_{\mathtt{j}}(0)| \geq 3^{-(n+1)}.$$

(ii) Determine a non-negative integer matrix A and a non-negative integer vector v for which there is a constant $c \geq 1$ so that

$$c^{-1} \|A^n v\|_1 \leq \#\{f_{\mathtt{i}} : \mathtt{i} \in \mathcal{I}^n\} \leq c \|A^n v\|_1 .$$

Note: depending on how you think about this counting problem, there are multiple natural choices for A. Probably, the matrix should be either 2×2 or 3×3.

(iii) Conclude that

$$\dim_{\mathrm{H}} K = \frac{\log\big((3+\sqrt{5})/2\big)}{\log 3}.$$

Hint: Use Gelfand's formula.