Exercise 4

DUE 12:15PM ON THURSDAY, FEBRUARY 5

The questions.

1. *(2 pt.)* Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system. Let $A \in \mathcal{B}$ have $\mu(A) > 0$. Prove that μ-a.e. $x \in A$ returns to A infinitely often (that is, $T^n x \in A$ for infinitely many n).

2. *(3 pt.)* Let (X, d) be a separable metric space, and let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving system where $\mathcal{B}$ is the Borel σ-algebra on X.
   (i) Prove for μ-a.e. $x \in A$ that there is a sequence of times $n_k \to \infty$ such that $\lim_{k\to\infty} T^{n_k} x = x$.
   (ii) Suppose moreover that the system is ergodic and every ball $B(x, r)$ has positive measure. Prove (without using the ergodic theorem) that μ-a.e. $x \in A$ has a dense orbit in X.

3. *(1 pt.)* Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f \colon X \to X$ be a measurable function. Prove that the family of sets

$$\mathcal{D} = \{E \in \mathcal{B} : \mu(f^{-1}(E)) = \mu(E)\}$$

   forms a *Dynkin class*. That is, it satisfies the following conditions:
   (a) $\varnothing \in \mathcal{D}$.
   (b) If $E \in \mathcal{D}$, then $X \setminus E \in \mathcal{D}$.
   (c) If $(E_n)_{n=1}^{\infty} \subset \mathcal{D}$ is a sequence of pairwise disjoint sets, then

$$\bigcup_{n=1}^{\infty} E_n \in \mathcal{D}.$$

4. *(4 pt.)* Let $\mathcal{I}$ be a finite index set and consider the infinite product space $\mathcal{I}^{\mathbb{N}}$. Let $\boldsymbol{p}^{\mathbb{N}}$ denote the infinite product measure, where $\boldsymbol{p} \in \mathcal{P}(\mathcal{I})$, and let $\sigma \colon \mathcal{I}^{\mathbb{N}} \to \mathcal{I}^{\mathbb{N}}$ denote the left shift map. Let

$$\mathcal{C} := \{[\mathtt{i}] : \mathtt{i} \in \mathcal{I}^*\}$$

   denote the set of cylinders, where we recall that

$$[\mathtt{i}] = \{x \in \mathcal{I}^{\mathbb{N}} : \mathtt{i} \prec x\}$$

   and $\mathtt{i} \prec x$ if $\mathtt{i} = (i_1, \ldots, i_n)$ and $x = (i_1, \ldots, i_n, j_{n+1}, j_{n+2}, \ldots,)$ for some $j_{n+k} \in \mathcal{I}$.

   (Recall that the infinite product measure is the unique probability measure defined on the σ-algebra generated by the cylinders $\mathcal{C}$ by the rule $\boldsymbol{p}^{\mathbb{N}}([\mathtt{i}]) = p_{\mathtt{i}}$.)

(i) Let $(r_i)_{i\in\mathcal{I}}$ be a set of numbers with $r_i \in (0,1)$. Given $x, y \in \mathcal{I}^{\mathbb{N}}$, define

$$d(x,y) = \inf\bigl\{r_{\mathtt{i}} : \{x,y\} \subset [\mathtt{i}]\bigr\}.$$

Prove that d defines a metric on $\mathcal{I}^{\mathbb{N}}$, and in fact satisfies the stronger *ultrametric inequality*:

$$d(x,z) \le \max\{d(x,y), d(y,z)\} \quad \text{for all} \quad x,y,z \in \mathcal{I}^{\mathbb{N}}.$$

(ii) Prove that $B^\circ(x,r) \in \mathcal{C}$ for all $x \in \mathcal{I}^{\mathbb{N}}$ and $r > 0$.

(iii) Conclude that $\mathcal{I}^{\mathbb{N}}$ is a compact metric space.
Hint: You may use Tychonoff's theorem to save some work, or it's not so much work to do this manually in any case.

(iv) Prove that if $[\mathtt{i}] \in \mathcal{C}$, then $\boldsymbol{p}^{\mathbb{N}}(\sigma^{-1}([\mathtt{i}])) = \boldsymbol{p}^{\mathbb{N}}([\mathtt{i}])$.

(v) Recall the π-λ theorem: *If $\mathcal{C}$ is a non-empty family of sets closed under finite intersection, $\mathcal{D}$ is a Dynkin class of sets, and $\mathcal{C} \subseteq \mathcal{D}$, then the σ-algebra generated by $\mathcal{C}$ is a subset of $\mathcal{D}$.*
Conclude that $\boldsymbol{p}^{\mathbb{N}}(\sigma^{-1}(E)) = \boldsymbol{p}^{\mathbb{N}}(E)$ for all Borel sets E.

(vi) **(1 pt. bonus)** Assume $\#\mathcal{I} \ge 2$. Give an example of a shift-invariant Borel probability measure on $\mathcal{I}^{\mathbb{N}}$ which is *not* a convex combination of the product measures $\boldsymbol{p}^{\mathbb{N}}$ for $\boldsymbol{p} \in \mathcal{P}(\mathcal{I})$.