Exercise 1 Solutions

THURSDAY, JANUARY 15

1. (i) Suppose $r > 0$ is fixed and let $x \in B(z, r)$. Let f_i have contraction ratio r_i. Then

$$d(f_i(x), z) \le d(f_i(x), f_i(z)) + d(f_i(z), z) \le r_i r + d(f_i(z), z).$$

So, we just need to choose r sufficiently large so that $r_i r + d(f_i(z), z) \le r$ for all $i \in \mathcal{I}$. For example, the choice $R = \max_{i \in \mathcal{I}} d(f_i(z), z)/(1 - r_i)$ is sufficient.

(ii) Let $\Phi(E) = \bigcup_{i \in \mathcal{I}} f_i(E)$. Observe that

$$\Phi(K_n) = \bigcup_{\mathtt{i} \in \mathcal{I}^n} f_{\mathtt{i}}(\Phi(Q)) = \bigcup_{\mathtt{i} \in \mathcal{I}^n} \bigcup_{i \in \mathcal{I}} f_{\mathtt{i}}(f_i(Q)) = K_{n+1}.$$

Moreover, since $\Phi(Q) \subset Q$, we see from the first equality that $\Phi(K_n) \subset K_n$.

(iii) By the Banach contraction mapping principle, $K = \lim_{n \to \infty} \Phi^n(Q)$. Since $\Phi^n(Q) \subset Q$, it follows that $K \subset Q$ and therefore $K \subset K_n$ for all $n \in \mathbb{N}$. Conversely, by definition of convergence in the Hausdorff metric, $\Phi^n(Q) \subset K^{(\varepsilon)}$ for all $\varepsilon > 0$ and n sufficiently large depending on ε. Therefore $\bigcap_{n=1}^\infty K_n \subset K^{(\varepsilon)}$ for all $\varepsilon > 0$, and therefore $\bigcap_{n=1}^\infty K_n = K$.

2. (i) Recall that $\Phi(E) = \bigcup_{i \in \mathcal{I}} f_i(E)$ is a contraction map, say with contraction ratio r. Moreover, an easy computation shows that $d_{\mathcal{H}}(A \cup F, B \cup F) \le d_{\mathcal{H}}(A, B)$. Therefore

$$d_{\mathcal{H}}(\Phi(A) \cap F, \Phi(B) \cap F) \le d_{\mathcal{H}}(\Phi(A), \Phi(B)) \le r d_{\mathcal{H}}(A, B).$$

Thus $E \mapsto \Phi(E) \cup F$ is a contraction map on $\mathcal{K}(X)$, so it has a unique fixed point K_F which is the claimed set.

(ii) Write $\Psi(E) = \bigcup_{i \in \mathcal{I}} f_i(E) \cup F$.
First, observe that $F^{(n)} = \Psi^n(F)$:

$$\Psi(F^{(n)}) = \bigcup_{k=0}^{n} \bigcup_{\mathtt{i} \in \mathcal{I}^k} \bigcup_{j \in \mathcal{I}} f_{\mathtt{i}j}(F) \cup F = F \cup \bigcup_{k=1}^{n+1} \bigcup_{\mathtt{i} \in \mathcal{I}^k} f_{\mathtt{i}}(F) = F^{(n+1)}.$$

Since $F \subset K_F$, it follows that $F^{(n)} = \Psi^n(F) \subset K_F$ for all n. Moreover, by the Banach contraction mapping principle, it follows that $K_F = \lim_{n \to \infty} F^{(n)}$.

(iii) This is $K_\varnothing$ (unless $F = \varnothing$, in which case it is $\varnothing$).

(iv) If $K_F = K_\varnothing$, then $F \subseteq K_F \subset K_\varnothing$. If $F \subset K_\varnothing$, then

$$K_\varnothing = K_\varnothing \cup F = \bigcup_{i \in \mathcal{I}} f_i(K_\varnothing) \cup F.$$

By uniqueness of K_F, it follows that $K_F = K_\varnothing$.

3. (i) Let Φ denote the Cantor IFS. There are a lot of choices here. For example, you could take any fixed starting point x, and take the orbit

$$E = \bigcup_{n=0}^{\infty} \Phi^n(\{x\}).$$

This is an invariant set by construction, and moreover E is a countable set and therefore not all of C.
If using uncountability of C feels bothersome, another option is to take $C \setminus \{0\}$. Since $0 \notin \Phi(\{x\})$ for all $x \neq 0$, $C \setminus \{0\}$ is invariant.

(ii) One option is to take $\mathbb{R}$, or another is to take $[0, \infty)$ for a proper closed subset. A more interesting option is to take $\bigcup_{n=0}^{\infty}\{3^n x : x \in C\}$. Check that this is actually invariant!

(iii) First, E must contain a non-zero point: if $0 \in E$, then its image under the second map is $2/3$, which must be in E. But then if $x \neq 0$, $x3^{-n} \in E$ for all $n \in \mathbb{N}$ is an infinite subset of E.

4. First, one can check that the map $g(x) = d(x, f(x))$ is 2-Lipschitz and therefore continuous. Since X is compact, there is an $x_* \in X$ which minimizes g.

If $x_* \neq f(x_*)$, then $d(f(x_*), f(f(x_*))) < d(x_*, f(x_*)) = g(x_*)$ contradicting minimality of $g(x_*)$. Moreover, if z is another fixed point of f and $z \neq x_*$, then $d(x_*, z) = d(f(x_*), f(z)) < d(x_*, z)$, which is a contradiction. Therefore x_* is the unique fixed point of f.

Finally, let $x_0 \in X$ be arbitrary and write $x_n = f^n(x_0)$ for $n \in \mathbb{N}$. First, observe that $d(x_*, x_{n+1}) = d(x_*, f(x_n)) \leq d(x_*, x_n)$. Therefore the sequence $a_n := d(x_*, x_n)$ is a decreasing sequence (bounded below by 0) and has some limit α.

Since X is compact, to show that $x_* = \lim_{n\to\infty} x_n$, it suffices to show that every accumulation point of (x_n) is x_*. Thus suppose $z = \lim_{k\to\infty} x_{n_k}$ is the limit of some subsequence. Note that $d(x_*, z) = \lim_{k\to\infty} d(x_*, x_{n_k}) = \alpha$. Moreover, since f is continuous,

$$d(x_*, f(z)) = \lim_{n\to\infty} d(x_*, f(x_{n_k})) = \lim_{n\to\infty} a_{n_k+1} = \alpha.$$

But if $z \neq x_*$, then $\alpha = d(x_*, f(z)) < d(x_*, z) = \alpha$, which is a contradiction. Therefore $z = x_*$, as required.