Exercise 2 Solutions

THURSDAY, JANUARY 22

1. (i) We first observe in the definition of the Hausdorff content of a compact set K that it suffices to consider covers using finite families of open sets. By definition of the Hausdorff content, get a family of sets $\{E_i\}_{i=1}^{\infty}$ covering K such that

$$\sum_{i=1}^{\infty} (\operatorname{diam} E_i)^s \leq \mathcal{H}^s_\infty(K) + \varepsilon.$$

For each E_i, let $\delta_i > 0$ be sufficiently small so that $(\operatorname{diam} E_i + 2\delta_i)^s \leq (\operatorname{diam} E_i)^s + \varepsilon 2^{-i}$. Then, for each E_i, consider the open neighbourhood $V_i = E_i^{(\delta_i)}$. Then $\{V_i\}_{i=1}^{\infty}$ is an open cover for K and therefore has a finite sub-cover, say $\{V_{i_1}, \ldots, V_{i_k}\}$. Observe that

$$\sum_{n=1}^{k} (\operatorname{diam} V_{i_n})^s \leq \sum_{n=1}^{k} (\operatorname{diam} E_i + 2\delta_i)^s \leq \mathcal{H}^s_\infty(K) + 2\varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, this completes the proof of this observation. Now, let $(K_n)_{n=1}^{\infty}$ be a sequence of compact sets with $K = \lim_{n\to\infty} K_n$. Write $c = \mathcal{H}^s_\infty(K)$ and let $\varepsilon > 0$. By the above observation, get a finite family of open sets $\{V_1, \ldots, V_k\}$ covering K such that

$$\sum_{i=1}^{k} (\operatorname{diam} V_i)^s \leq c + \varepsilon.$$

Now consider the new sets $W_i = V_i^{(\eta)}$ for some $\eta > 0$. Since the sets V_i cover K, the sets W_i cover $K^{(\eta)}$. Therefore, for all n sufficiently large depending on η, $K_n \subset \bigcup_{i=1}^{k} W_i$ so that

$$\mathcal{H}^s_\infty(K_n) \leq \sum_{i=1}^{k} (\operatorname{diam} W_i)^s \leq \sum_{i=1}^{k} (\operatorname{diam} V_i + 2\eta)^s.$$

Since k is fixed (independently of n) and $\eta > 0$ is arbitrary,

$$\limsup_{n\to\infty} \mathcal{H}^s_\infty(K_n) \leq \sum_{i=1}^{k} (\operatorname{diam} V_i)^s \leq c + \varepsilon.$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

(ii) Let $K_n = \{j/N : 0 \leq j \leq N\}$. Then $\mathcal{H}^{1/2}(K_n) = 0$ for all n, whereas $\lim_{n\to\infty} K_n = [0,1]$ has $\mathcal{H}^{1/2}([0,1]) > 0$.

(iii) Let $K_n = [0, 1/n]$, so $\mathcal{H}^{1/2}(K_n) = \infty$. However, $\lim_{n\to\infty} K_n = \{0\}$ and which has $1/2$-dimensional Hausdorff measure 0.

2. (i) If $\{B(x_i, 2^{-u})\}_i$ is a cover for E, then $\{B(x_i, 2^{-v})\}$ is also a cover for E. Therefore $f_E(u) \geq f_E(v)$. For the other inequality, note that any ball of radius 2^{-v} can be covered by $2 \cdot 2^{u-v}$ balls of radius 2^{-u}. Any such ball of radius 2^{-u} which intersects E can in turn be covered by 2 balls of radius 2^{-u} centred in E. Therefore

$$N_{2^{-u}}(E) \leq 4 \cdot 2^{u-v} N_{2^{-v}}(E).$$

Taking logarithms and rearranging, the conclusion follows.

(ii) The analogue is the following: there is a constant $M_d \geq 0$ so that for $v \leq u$,

$$0 \leq f_E(u) - f_E(v) \leq d(u-v) + M.$$

(iii) Suppose $2^{-n} \leq r < 2^{-n+1}$ and let $B(x,r)$ be an arbitrary ball. Then $B(x,r)$ intersects at most 5 dyadic intervals of side-length 2^{-n}. Conversely, any interval of side-length 2^{-n} is contained in any ball $B(x,r)$ where x is in the interval. Therefore

$$\Delta_r(E) \leq N_r(E) \leq 5\Delta_r(E).$$

(iv) Let $r > 0$ and let $\{B(x_i, r)\}_{i=1}^m$ be a cover for E with $m = N_r(E)$. Then $\{B(x_i, 2r)\}_{i=1}^m$ is a cover for $E^{(r)}$ so

$$m(E^{(r)}) \leq 2r \cdot N_r(E).$$

Conversely, let $\{y_i\}_{i=1}^k$ be a maximal r-separated subset of E, so that the Lebesgue measure of the intersections of the balls $B(y_i, r/2)$ are 0. By maximality, $\{B(y_i, r)\}_{i=1}^k$ is a cover for E, so $N_r(E) \leq k$. Moreover, $B(y_i, r/2) \subset E^{(r)}$ for all i, so

$$m(E^{(r)}) \geq \sum_{i=1}^k m(B(y_i, r/2)) = kr \geq rN_r(E)$$

This completes the proof.

3. (i) Observe that K_n is a union of $2^{\sum_{n=1}^k a_n}$ dyadic intervals. Moreover, each dyadic interval in this union intersects K. Taking into account the endpoints of the intervals, it follows that

$$2^{\sum_{n=1}^k a_n} \leq \Delta_r(K) \leq 3 \cdot 2^{\sum_{n=1}^k a_n}.$$

Thus the claim follows from Q2(iii).
To complete the proof, take logarithms, divide by $\log(1/r)$ and pass to the limit.

(ii) If $\underline{\dim}_{\mathrm{B}} K = 0$ there is nothing to prove, so we may assume otherwise. We will use the mass distribution principle. Let $\ell_k = 2^{\sum_{n=1}^{k} a_n}$ denote the number of distinct dyadic intervals in the construction of K_k. Using the method of subdivision, let μ be a measure supported on K such that $\mu(I) = \ell_k^{-1}$ for all dyadic intervals in the construction of I.
Let $0 < s < \underline{\dim}_{\mathrm{B}} K$ be arbitrary. Applying (i), there exists a constant $c > 0$ such that $\ell_k \geq c2^{ks}$ for all $k \in \mathbb{N}$. Now, suppose A is an arbitrary Borel set with $\operatorname{diam} A < 1$, and let $k \in \mathbb{N}$ be such that $2^{-k} \leq \operatorname{diam} A < 2^{-k+1}$. Then A intersects at most 4 dyadic intervals of side-length 2^{-k}, so

$$\mu(A) \leq 4\ell_k^{-1} \leq 4c^{-1}2^{ks} \leq 8c^{-1}(\operatorname{diam} A)^s.$$

Therefore μ is s-Frostman so $\dim_{\mathrm{H}} K \geq s$. Since $s < \underline{\dim}_{\mathrm{B}} K$ was arbitrary, it follows that $\dim_{\mathrm{H}} K = \underline{\dim}_{\mathrm{B}} K$.

(iii) It is enough to choose the sequence $(a_n)_{n=1}^{\infty} \in \{0,1\}^{\mathbb{N}}$ judiciously.

$$n\alpha - 1 < u_1 + \cdots + u_n \leq n\alpha \tag{1}$$

for all $n \in \mathbb{N}$. To start, we can take $u_1 = 0$. Now suppose we have chosen $(u_1, \ldots, u_n)$ satisfying (1). Then $(n+1)\alpha - 1 \leq n\alpha$ and $(n+1)\alpha \leq n\alpha + 1$, so we may choose $u_{n+1} \in \{0,1\}$ so that

$$(n+1)\alpha - 1 < u_1 + \cdots + u_n + u_{n+1} \leq (n+1)\alpha.$$

Now, let:
- $(u_n)_{n=1}^{\infty}$ satisfy (1) with $\alpha = s$.
- $(v_n)_{n=1}^{\infty}$ satisfy (1) with $\alpha = t$.

We inductively define a sequence $(a_n)_{n=1}^{\infty}$ with partial sums $s_n = a_1 + \cdots + a_n$ as follows.
Begin with $a_1 = v_1$ and $m_1 = 1$. Now, suppose we have defined $(a_n)_{n=1}^{m_n}$ for some $m_n \in \mathbb{N}$.

- If n is odd, by (1), we may choose $N \geq n$ sufficiently large so that

$$\frac{s_n + u_1 + \cdots + u_N}{n+N} \leq s + \frac{1}{n}. \tag{2}$$

  Then set $a_{n+j} = u_j$ for $j = 1, \ldots, N$ and let $m_{n+1} = m_n + N$.
- If n is even, by (1), we may choose $N \geq n$ sufficiently large so that

$$t - \frac{1}{n} \leq \frac{s_n + v_1 + \cdots + v_N}{n+N}. \tag{3}$$

  Then set $a_{n+j} = v_j$ for $j = 1, \ldots, N$ and let $m_{n+1} = m_n + N$.

The choice (2) ensures that $\liminf_{n\to\infty} s_n \leq s$ and the choice (3) ensures that $\limsup_{n\to\infty} s_n \geq t$.
It remains to verify the other inequalities. For $n \in \mathbb{N}$, we may write

$$s_n = u_1 + \cdots + u_{m_1} + v_1 + \cdots + v_{m_2} + \cdots + v_1 + \cdots + v_{m_k} + v_1 + \cdots + v_\ell.$$

(The final term may consist instead of terms $v_1, \ldots, v_\ell$, but the argument is the same in that case.) Then using (1),

$$s_n \le m_1 s + m_2 t + m_3 s + \cdots + m_k t + \ell s \le nt.$$

Therefore $\limsup_{n\to\infty} n^{-1} s_n \le t$. For the lower bound,

$$s_n \ge (m_1 s - 1) + (m_2 t - 1) + \cdots + (m_k t - 1) + \ell s - 1 \ge ns - k - 1.$$

But $m_{k+1} \ge m_k + k$ by the choice of N, so $n \ge k(k+1)/2$ and

$$\liminf_{n\to\infty} n^{-1} s_n \ge s$$

as required.

(iv) Now, let us give an alternative proof of the previous question using 1-Lipschitz functions, and moreover prove the bonus.
Let's first show that we may equivalently choose an appropriate Lipschitz function. More precisely, we prove the following: *Suppose f is an increasing 1-Lipschitz function with $f(0) = 0$. Then there exists a sequence $(a_n)_{n=1}^\infty \in \{0,1\}^{\mathbb{N}}$ such that $f(k) - 1 < \sum_{n=1}^k a_n \le f(k)$ for all $k \in \mathbb{N}$.*
The construction proceeds by induction. Let $s_n = a_1 + \cdots + a_n$ denote the partial sum, where $s_0 = 0$. Suppose we have constructed $(a_n)_{n=1}^k$ such that $f(k) - 1 < s_k \le f(k)$.

- If $s_k + 1 \le f(k+1)$, set $a_{k+1} = 1$, and note that $s_{k+1} \le f(k+1)$ by assumption, and since f is 1-Lipschitz, $f(k+1) \le f(k) + 1 < s_k + 2 = s_{k+1} + 1$.
- Otherwise, if $s_k + 1 > f(k+1)$, set $a_{k+1} = 0$. Then $s_{k+1} > f(k+1) - 1$ by assumption, and since f is increasing, $s_{k+1} = s_k \le f(k) \le f(k+1)$.

This completes the proof of the claim.
Given a function f satisfying the above properties, if $(a_n)_{n=1}^\infty$ is the associated sequence, we write $K_f = K(a_n)_{n=1}^\infty$. In particular, we may check that there is a constant $M \ge 0$ so that for all $u \ge 0$,

$$\left| \frac{\log N_{2^{-u}}(K_f)}{\log 2} - f(u) \right| \le M.$$

(This is immediate if u is an integer, and if $n \le u \le n+1$ use the fact that $f(n) \le f(u) \le f(n) + 1$.) Therefore,

$$\underline{\dim}_{\mathrm{B}} K_f = \liminf_{u\to\infty} \frac{f(u)}{u},$$
$$\overline{\dim}_{\mathrm{B}} K_f = \limsup_{u\to\infty} \frac{f(u)}{u}.$$

Let $0 \le s \le t \le 1$ be arbitrary. We define a 1-Lipschitz function g as follows. Suppose we have chosen values $0 = x_1 < y_1 < x_2 < y_2 < \cdots$

diverging to infinity. Having chosen such values, we define g to be the unique Lipschitz function such $g(0) = 0$, g has slope s on the intervals $[x_i, y_i]$, and g has slope t on the intervals $[y_i, x_{i+1}]$. Clearly $su \leq g(u) \leq tu$ regardless of the choice of the x_i and y_i. Moreover, for all $n \in \mathbb{N}$,

$$
\begin{aligned}
\frac{g(y_n)}{y_n} &= \frac{g(x_n) + s(y_n - x_n)}{y_n} \leq s + t\frac{x_n}{y_n}, \\
\frac{g(x_{n+1})}{x_{n+1}} &= \frac{g(y_n) + t(x_{n+1} - y_n)}{x_{n+1}} \geq t - t\frac{y_n}{x_{n+1}}.
\end{aligned} \tag{4}
$$

Therefore, if we choose the x_n and y_n such that $\lim_{n\to\infty} x_n/y_n = 0$ and $\lim_{n\to\infty} y_n/x_{n+1}n = 0$ it follows that $\liminf_{n\to\infty} g(u)/u = s$ and $\limsup_{n\to\infty} g(u)/u = t$. Thus the corresponding set K satisfies the required properties.

Next, let us turn our attention to the actual bonus problem. We must construct two increasing 1-Lipschitz functions f and g with $f(0) = g(0) = 0$. Let K_f and K_g denote the corresponding sets. The heart of the strategy is the following straightforward observation:

$$
\left| \frac{\log N_{2^{-u}}(K_f \cup K_g)}{\log 2} - \max\{f(u), g(u)\} \right| \leq 2M.
$$

So, it suffices to choose the functions f and g with the following properties:

(i) $\liminf_{u\to\infty} u^{-1} f(u) = \liminf_{u\to\infty} u^{-1} g(u) = s$, and
(ii) $\lim_{u\to\infty} u^{-1} \max\{f(u), g(u)\} = t$.

The first condition guarantees that $\underline{\dim}_{\mathrm{B}} K_f = \underline{\dim}_{\mathrm{B}} K_g = s$ and the second condition guarantees that $\dim_{\mathrm{B}}(K_f \cup K_g) = t$.

Suppose we have chosen sequences $0 = w_1 < x_1 < y_1 < z_1 < w_2 < x_2 < y_2 < z_2 < \cdots$ diverging to infinity. Define the function f to have slope s on the intervals $[x_i, y_i]$ and slope t on the remaining intervals, and define the function g to slope s on the intervals $[z_i, w_{i+1}]$ and slope t on the remaining intervals. The point is that whenever f has slope s on some interval, g has slope t on that interval as well as on the preceding and following interval; and similarly with f and g swapped. Similarly to before, we suppose moreover that the gaps diverge:

$$
\lim_{n\to\infty} \frac{w_n}{x_n} = \lim_{n\to\infty} \frac{x_n}{y_n} = \lim_{n\to\infty} \frac{y_n}{z_n} = \lim_{n\to\infty} \frac{z_n}{w_{n+1}} = 0
$$

The computation from (4) already shows that (i) holds. Moreover, the computation from (4) shows for $u \in [z_n, x_{n+1}]$ that

$$
\frac{f(u)}{u} \geq t - t\frac{y_n}{z_n}
$$

and, for $u \in [x_{n+1}, z_{n+1}]$ that

$$
\frac{g(u)}{u} \geq t - t\frac{z_n}{x_{n+1}}.
$$

This establishes (ii), as required.

4. (i) Let $\mathtt{j} \neq \mathtt{k}$ be such that $f_{\mathtt{j}} = f_{\mathtt{k}}$. Then $\mathtt{jk} \neq \mathtt{kj}$ whereas $f_{\mathtt{jk}} = f_{\mathtt{kj}}$.

   (ii) Now, let $\mathtt{j} \neq \mathtt{k}$ be such that $f_{\mathtt{j}} = f_{\mathtt{k}}$ and $\mathtt{j}, \mathtt{k} \in \mathcal{I}^n$. Iterating the invariance relationship,

$$K = \bigcup_{\mathtt{i} \in \mathcal{I}^n} f_{\mathtt{i}}(K) = \bigcup_{\mathtt{i} \in \mathcal{I}^n \setminus \{j\}} f_{\mathtt{i}}(K).$$

   Therefore, by uniqueness K is the attractor of the modified IFS Φ'.

   (iii) Let $s \geq 0$ be the unique solution to $\sum_{i \in \mathcal{I}} r_i^s = 1$. Then,

$$1 = \left(\sum_{i \in \mathcal{I}} r_i^s \right)^n = \sum_{\mathtt{i} \in \mathcal{I}^n} r_{\mathtt{i}}^s > \sum_{\mathtt{i} \in \mathcal{I}^n \setminus \{\mathtt{j}\}} r_{\mathtt{i}}^s$$

   since $r_{\mathtt{j}}^s > 0$. But the map

$$t \mapsto \sum_{\mathtt{i} \in \mathcal{I}^n \setminus \{\mathtt{j}\}} r_{\mathtt{i}}^t$$

   is strictly decreasing in t, so the similarity dimension of Φ' must be strictly less than s.