Oath Dice Odds

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ABSTRACT. We compute some odds related to the combat system in the Oath board game.

CONTENTS

1. OATH DICE ODDS

1.1. **Combat Mechanics.** Both the attacking and defending dice are standard 6-sided dice. The defence dice have the following states:

state	count
0	2
+1	2
+2	1
$\times 2$	1

The attack dice have the following states

state	count
+1/2	3
+1	2
+2	1

When defending, the counts are computed by first adding up all the $+1$ and $+2$ states, and then multiplying 2 raised to the power of the number of $\times 2$ dice. When attacking, the counts are computed by adding all the states, and then rounding down to the nearest integer. Then, after rolling the dice, the attacker must sacrifice a number of units equal to the number of $+2$ states.

1.2. **Defender dice odds.** Suppose the defender is rolling n dice. It is convenient to think of the outcomes as elements of $\Omega_n \cong \{1, \ldots, 6\}^n$ using the correspondence between sides of the dice and $\{1,\dots,6\}$. Then a defensive dice roll is the random variable $D:\Omega_n\to\{0,1,\dots,2^n\}$ given by the rule

$$D(a_1,\dots,a_n)=\big(\#\{i:a_i\in\{3,4\}\}+2\cdot\#\{i:a_i=5\}\big)\cdot 2^{\#\{i:a_i=6\}}$$

where $\#A$ denotes the cardinality of the set A. Let's compute the distribution of D, i.e. $\mathbb{P}(D=2^m\ell)$ where ℓ is odd. We obtain a value j precisely when there are k dice which show $\times 2$, and the remaining dice sum to exactly $2^{m-k}\ell$, for some $0\leq k\leq m$. There are $\binom{n}{k}$ ways to roll k dice showing $\times 2$. To compute the number of ways for the remaining dice to show precisely $2^{m-k}\ell$, we first observe that the number of ways for $n-k$ dice to sum to exactly $2^{m-k}\ell$ is the coefficient of $x^{2^{m-k}\ell}$ in the expansion of $(2+2x+x^2)^{n-k}$. Then, writing $2+2x+x^2=1+(1+x)^2$ and expanding the binomial twice,

$$\begin{aligned}(2+2x+x^2)^{n-k}&=\sum_{i=0}^{n-k}\binom{n-k}{i}(x+1)^{2i}\\&=\sum_{i=0}^{n-k}\binom{n-k}{i}\sum_{j=0}^{2i}\binom{2i}{j}x^j\end{aligned}$$

so that

$$[x^{2^{m-k}\ell}](2+2x+x^2)^{n-k}=\sum_{i=0}^{n-k}\binom{n-k}{i}\binom{2i}{2^{m-k}\ell}.$$

Since $\#\Omega_n=6^n$, we conclude that

$$\mathbb{P}(D=2^m\ell)=6^{-n}\sum_{k=0}^{m}\sum_{i=0}^{n-k}\binom{n}{k}\binom{n-k}{i}\binom{2i}{2^{m-k}\ell}.$$

1.3. **Attacker dice odds.** Now, let's compute the attacker dice odds, which are somewhat more straightforward but have some somewhat annoying rounding cases that we need to deal with. First, define a random variable $A_0:\Omega_n\to\{0,1/2,1,\dots,2n\}$ given by the rule

$$A_0(a_1,\dots,a_n)=\#\{i:a_i\in\{0,1,2\}\}/2+\#\{i:a_i\in 3,4\}+2\cdot\#\{i:a_i=6\}.$$

so that $A:=\lfloor A_0\rfloor:\Omega_n\to\{0,1,\dots,2n\}$ represents the attacking dice roll. In fact, it will be useful to also keep track of the number of lost units from the dice roll as well. Let $S:\Omega_n\to\{0,\dots,n\}$ denote the number of units lost for a $+2$ on an attacking dice:

$$S(a_1,\dots,a_n)=\#\{i:a_i=6\}.$$

We can use the same trick as before to compute the joint distribution of A_0 and S: $\mathbb{P}(A_0=z\text{ and }S=k)$ is precisely the coefficient of x^zy^k in the expansion of

$(3x^{1/2}+2x+x^2y)^n$. Expanding the binomial twice, we have

$$\begin{aligned}(3x^{1/2}+2x+x^2y)^n &= \sum_{k=0}^{n}\binom{n}{k}x^k(2+xy)^k(3x^{1/2})^{n-k}\\ &= \sum_{k=0}^{n}\binom{n}{k}3^{n-k}x^{(n+k)/2}\sum_{j=0}^{k}\binom{k}{j}(xy)^j2^{k-j}\\ &= \sum_{k=0}^{n}\sum_{j=0}^{k}\binom{n}{k}\binom{k}{j}3^{n-k}2^{k-j}x^{j+(n+k)/2}y^j.\end{aligned}$$

First, let's compute the coefficients of $[x^j]$ in the above scenarios. Write $p_n(x,y)=(3x^{1/2}+2x+x^2y)^n$. Re-indexing to sum over even or odd values of k, we have the following four cases:

(a) If $n=2m$ and $j\in\{0,\ldots,2n\}$, then

$$[x^j]p_n(x,j)=\sum_{k=0}^{m}\binom{n}{2k}\binom{2k}{j-m-k}3^{n-2k}2^{3k+m-j}y^{j-m-k}.$$

(b) If $n=2m$ and $j\in\{0,\ldots,2n-1\}$, then

$$[x^{j+1/2}]p_n(x,j)=\sum_{k=0}^{m}\binom{n}{2k+1}\binom{2k+1}{j-m-k}3^{n-2k-1}2^{(3k+1)+m-j}y^{j-m-k}.$$

(c) If $n=2m+1$ and $j\in\{0,\ldots,2n\}$, then

$$[x^j]p_n(x,j)=\sum_{k=0}^{m}\binom{n}{2k+1}\binom{2k+1}{j-m-k-1}3^{n-2k-1}2^{(3k+2)+m-j}y^{j-m-k-1}.$$

(d) If $n=2m+1$ and $j\in\{0,\ldots,2n-1\}$, then

$$[x^{j+1/2}]p_n(x,j)=\sum_{k=0}^{m}\binom{n}{2k}\binom{2k}{j-m-k}3^{n-2k}2^{3k+m-j}y^{j-m-k}.$$

Now, substituting $y=1$ into the above equations, if $j\in\{0,\ldots,2n-1\}$ and $n=2m$,

$$\begin{aligned}\mathbb{P}(A=j)=\sum_{k=0}^{m}3^{-2k}2^{3k-m-j}\Bigg(&\binom{n}{2k}\binom{2k}{j-m-k}\\ &+\frac{2}{3}\binom{n}{2k+1}\binom{2k+1}{j-m-k}\Bigg)\end{aligned}$$

and if $j \in \{0, \dots, 2n-1\}$ and $n = 2m+1$,

$$\mathbb{P}(A = j) = \sum_{k=0}^{m} 3^{-2k} 2^{3k-m-j} \left(\frac{2}{3} \binom{n}{2k+1} \binom{2k+1}{j-m-k-1} + \frac{1}{2} \binom{n}{2k} \binom{2k}{j-m-k} \right)$$

and

$$\mathbb{P}(A = 2n) = 6^{-n}$$

since the only way to roll $2n$ is for every dice to roll $+2$.

Moreover, the joint distributions can be extracted by simultaneously capturing the coefficient of x and y. Let $\ell \in \{0, \dots, 2n\}$. If $n = 2m$ and $j \in \{0, \dots, 2n-1\}$,

$$\mathbb{P}((A, S) = (j, l)) = 3^{-2(j-m-\ell)} 2^{2j-4m-3\ell} \left(\binom{n}{2(j-m-\ell)} \binom{2(j-m-\ell)}{\ell} + \frac{2}{3} \binom{n}{2(j-m-\ell)+1} \binom{2(j-m-\ell)+1}{\ell} \right)$$

and if $n = 2m+1$ and $j \in \{0, \dots, 2n-1\}$,

$$\mathbb{P}((A, S) = (j, l)) = 3^{-2(j-m-\ell)} 2^{2j-4m-3\ell-1} \left(\frac{3}{2} \binom{n}{2(j-m-l)-1} \binom{2(j-m-l)-1}{\ell} + \binom{n}{2(j-m-\ell)} \binom{2(j-m-\ell)}{\ell} \right)$$

and finally if $j = 2n$, then we must have $\ell = n$ so

$$\mathbb{P}((A, S) = (2n, n)) = 6^{-n}$$

and $\mathbb{P}((A, S) = (2n, \ell)) = 0$ for $\ell \neq n$.

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